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THE SHAPE OF A PENDENT LIQUID DROP

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CONTENTS

A characterization is given for the most general equilibrium configuration of a symmetric pendent liquid drop. It is shown that for any vertex height u_0 the vertical section can be continued globally as an analytic curve, without limit sets or double points. For small $|u_0|$ it is proved the section projects simply on the axis $u = 0$; for large $|u_0|$ the section is shown to have near the vertex the general form of a succession of circular arcs joined near the axis by small arcs of large curvature. The section contracts at first toward a certain hyperbola, the 'circular arcs' gradually changing shape but remaining, until a certain fixed height (asymptotically as $u_0 \to -\infty$), within a narrow band surrounding the hyperbola. The continuation of the section eventually projects simply on $u = 0$, separates from the hyperbola, and continues in an oscillatory manner to infinity. The properties described above are studied quantitatively.

It is conjectured that as $|u_0| \to \infty$ the section converges uniformly (as a point set) to a solution $U(r)$ with simple projection (for all $r > 0$) on $u = 0$ and an isolated singularity at $r = 0$. A preliminary (weak) form of the conjecture is proved.

The (liquid drop) solutions are also studied from the point of view of their global embedding in the manifold of all formal solutions of the equations. From this point of view, the vertex of the drop appears as a transition point marking a change in qualitative appearance. It is conjectured that the only global solution without double points, in this extended sense, is the singular solution referred to above.

Introduction

The form of the outer surface of a pendent liquid drop is determined by the condition that the mean curvature of the surface be proportional to the distance below a horizontal reference plane. For points on the surface at which the height can be described by a function $z = u(\boldsymbol{x})$ we obtain an equation

$$\operatorname{div}\left\{\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}}\right\} = \kappa u + \lambda, \tag{1}$$

where u is measured positively upward from the plane; κ and λ are constants. We consider in this paper the case of physical interest $\boldsymbol{x} = (x_1, x_2)$, although the methods will extend without essential change to any number of variables. For background information on the derivation of (1) see, for example, Minkowski (1907), Bakker (1928), Finn (1974).

In (1) κ is a physical constant, $\kappa < 0$ when the liquid lies above the surface, and λ is a Lagrange parameter, to be determined by the constraints. (For example, one might constrain the drop to have a prescribed volume.) In a specific problem the determination of λ may lead to technical difficulties. Formally, however, λ can be transformed out of (1) by adding a constant to u. In the present paper, we shall characterize all symmetric solutions for the case $\lambda = 0$. A solution corresponding to a given λ can then be found in this family by transforming back.

We restrict attention throughout this paper to rotationally symmetric configurations, such as would occur for a drop suspended from a circular aperture. Introducing the (inessential and convenient) normalization $\kappa = -1$ we obtain, in terms of polar radius r, the equation

$$\left(\frac{r u_r}{(1+u_r^2)^{\frac{1}{2}}}\right)_r = -ru \tag{2}$$

for a symmetric two dimensional surface $u(r)$. The subscript r denotes differentiation with respect to r.

Not all surfaces that appear physically have a simple projection on a base plane, hence for a complete description the form (2) is overly restrictive. We obtain a more suitable (parametric) form of the problem if we introduce the arc length s along a vertical section of the interfacial surface, measured from the vertex $(0, u_0)$. We are led to the system

$$\left.\begin{aligned} \mathrm{d}\psi/\mathrm{d}s &= -u - (\sin\psi)/r, \\ \mathrm{d}u/\mathrm{d}s &= \sin\psi, \\ \mathrm{d}r/\mathrm{d}s &= \cos\psi, \end{aligned}\right\} \tag{3}$$

where ψ is the angle between a tangent to the section and the r axis, measured counterclockwise from the positively directed axis to the tangent line.

From the point of view of general theory, one would expect a solution of (3) to be determined, at least locally, by the initial data

$$r(0) = 0; \quad \psi(0) = 0; \quad u(0) = u_0; \tag{4}$$

however, the system (3) is singular at $s = 0$, and because of this the second condition in (4) is superfluous (cf. the discussion in Concus & Finn 1975 *b*).

The question of local existence in a neighbourhood of a singular point has been studied by Lohnstein (1891), who established the convergence of a formal power series expansion. We have been unable to locate Lohnstein's paper and have had to infer its contents from the general reports

(Minkowski 1907; Bakker 1928). Presumably the Picard method, as adapted by Johnson & Perko (1968) for the capillarity problem with $\kappa > 0$, could be made to work also in the case studied here. For the convenience of the reader, we present in the appendix another proof – based on the Schauder fixed point theorem – which seems to us conceptually more accessible.

One obtains locally near $r = 0$, by these methods, a non-parametric solution $u(r)$ of the equation (2), which we may write in the form

$$(r \sin \psi)_r = -ru, \tag{5}$$

corresponding to the (single) initial condition

$$u(0) = u_0. \tag{6}$$

This solution defines the meridional curve of a capillary rotation surface, with (regular) vertex on the axis $r = 0$.

The circumstance that only one initial datum is required yields an important simplification for the problem of characterizing all solutions. It suffices to describe the one-parameter family determined by u_0, and it is this approach we adopt in the present work.

In general, the solution $u(r; u_0)$ determined in this way cannot be continued indefinitely as solution of (5). We shall show however that *for any u_0, the function $u(r; u_0)$ can be extended as a parametric solution of (3) for all s, yielding a curve without limit sets or double points. The resulting capillary rotation surface does not again contact the axis $r = 0$, and in fact spreads out indefinitely away from that axis.*

We shall characterize quantitatively the asymptotic form of the surface in the case of large $|u_0|$, and we shall characterize qualitatively the global structure of all such surfaces.

The global behaviour changes qualitatively when $|u_0|$ increases beyond a critical value. If $|u_0|$ is large, there is an initial range for s in which the surface looks like a succession of spheres centered on the u axis with radius $\approx (2/|u|)$. In all cases, however, the section can be expressed for large s in the form $u(r)$ and has an oscillatory behaviour as $r \to \infty$.

If $u_0 = 0$ the unique solution of (2) is given by $u \equiv 0$. We assume throughout this paper that $u_0 < 0$; the remaining case is obtained by a simple change of sign.† We are interested particularly in what happens when u_0 is large negative. The resulting surfaces are then physically unstable under most conditions of everyday experience; however, the problem has an independent mathematical interest (one specific feature of which we indicate below) and probably also a physical interest for situations in which gravity forces are small compared with those of surface tension.

We have proved (in Concus & Finn 1975 *a*) the existence of a particular singular solution of (2) that can be expressed in the form $U(r)$ in $0 < r < \delta$, and such that $U(r) \simeq -r^{-1}$ as $r \to 0$. In an earlier paper (Concus & Finn 1974) we have presented numerical evidence suggesting that the symmetric solutions discussed above tend uniformly (as point sets) to $U(r)$ as $|u_0| \to \infty$. A particular consequence of the analysis in the present paper will be a proof of a preliminary form of that conjecture, namely we shall show in § VI and § VII that the solutions converge asymptotically into a neighbourhood of $U(r)$, the size of which can be estimated *a priori* and is small relative to other (local) distances. The considerations in these sections are rather technical, and the reader who wishes only a general idea of the method and of its possibilities may find them in the earlier sections of the paper.

† The remaining case can be realized physically, e.g. as the lower surface of a column of water in a glass capillary tube.

In § VIII we point out a compactness property, again suggestive for our conjecture. In § IX the structure of solutions is examined from another point of view and some numerical results are presented, suggesting the types of possible global behaviour. These results lead in turn to another conjecture, namely that the singular solution $U(r)$ is the only global solution (in an extended sense) that is free of double points.

We remark that we know of few other studies of the problem from a general theoretical point of view.† To our knowledge the first attempt to characterize the shape of a liquid drop appears in a paper by Bashforth & Adams (1883), in which a numerical procedure is developed and applied to configurations in which a vertical point may appear. Thomson (1886) used a geometrical method and was able to obtain a figure corresponding, in our notation, to $u_0 \approx -7$. Computational studies were greatly facilitated by development of high speed computers and related techniques, and many more particular cases have now been calculated (see, for example, Hida & Miura 1970; Padday 1971; Concus & Finn 1974; Hartland & Hartley 1976, where also further references can be found). Such calculations are suggestive and instructive, but they cannot provide the unifying insight of a general formal description. The present work is intended as an initial step toward that objective.

In this work we study the formal solutions of the static equilibrium equations. We do not here treat the related question of stability; with regard to that matter, the reader may wish to consult the recent papers of Pitts (1973, 1974), Hida & Miura (1970), and also a new contribution by Wente (1978).

The central difficulty in the general study of the solutions of (2) lies in the failure of the maximum principle. We replace this principle here by a geometrical one, the conceptual content of which is that if surfaces S_1 and S_2 contact at p, and if their mean curvature vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ at p satisfy $\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 = \alpha\boldsymbol{\omega}_1$, $\alpha > 0$, then S_1 lies locally on the side of S_2 into which $\boldsymbol{\omega}_1$ is directed. The analytical formulation II i of this principle in the situation encountered here yields a global result and encompasses also the case $\alpha = 0$. These circumstances, in conjunction with formal manipulation of the equation, provide the central tools in our investigation. We proceed in a succession of steps, most of which are elementary and immediate; when taken together, however, they yield the requisite characterization.

We remark that the comparison technique just mentioned has proved effective also in other (related) contexts, and has led in particular to new information on the behaviour of solutions of (2) near isolated singular points (see, for example, Finn 1976).

I. The case of small $|u_0|$

We shall prove:

Theorem 1. *If, in the initial value problem* (5, 6) *there holds*‡ $u_0 \geqslant -2$, *then the solution can be continued as a (nonparametric) solution of the equation*

$$\left(\frac{ru_r}{(1+u_r^2)^{\frac{1}{2}}}\right)_r = -ru \tag{2}$$

for all $r > 0$. *It has an infinity of zeros. For any two successive extrema* r_a, r_b *of* $u(r)$ *there holds* $|u(r_b)| < |u(r_a)|$. *Asymptotically as* $u_0 \to 0$ *the first zero* r_0 *is the first zero of the Bessel function* $J_0(r)$, $r_0 \approx 2.405$.

† We call attention, however, to a remarkable existence theorem due to Wente (1973).
‡ This improves the result announced in Concus & Finn (1974).

We study first the portion of the trajectory preceding the first zero, and we note that (2) is equivalent to (5) on any interval on which $|u_r| < \infty$.

I i. *Let* $u(r)$ *satisfy* (5) *in* $0 < r < R$ *and* (6) *at* $r = 0$. *Then*† $\sin \Psi(0) = 0$.

Proof. Integrating (5) from $\epsilon > 0$ to r, we find

$$r \sin \psi - \epsilon \sin \psi(\epsilon) = -\int_{\epsilon}^{r} \rho u(\rho) \mathrm{d}\rho;$$

hence, using (6), we obtain

$$\sin \psi = \frac{u_r}{(1+u_r^2)^{\frac{1}{2}}} = -\frac{1}{r}\int_0^r \rho u(\rho)\, \mathrm{d}\rho, \tag{7}$$

from which we conclude $\lim_{r\to 0} u_r(r) = 0$. Hence there exists

$$u_r(0) = \lim_{r\to 0} \frac{u(r)-u_0}{r} = \lim_{r\to 0} \frac{1}{r}\int_0^r u_r(\rho)\, \mathrm{d}\rho = 0.$$

I ii. *Let* $u(r)$ *satisfy* (5) *in* $0 < r < R$ *and* (6) *at* $r = 0$. *If* $u(r) < 0$ *in* $0 < r < R$, *then* $\sin \psi > 0$ *in that interval.*

The proof is contained in (7).

It follows in particular that $u(r) \to u_R \leqslant 0$ as $r \to R$, that $\sin \psi_R = \lim_{r\to R} \sin \psi(r)$ exists, and that

$$0 < \sin \psi_R = -\frac{1}{R}\int_0^R \rho u(\rho)\, \mathrm{d}\rho \leqslant 1.$$

We conclude also that if the solution curve does not cross the hyperbola $ru = -1$, then $\sin \psi_R < 1$. The following assertion covers as well the case of solution curves crossing that hyperbola.

I iii. *Under the hypotheses of* I ii, *if in addition* $u_0 \geqslant -2$, *then* $0 < \sin \psi < 1$ *in* $0 < r \leqslant R$.

Proof. Consider the relation

$$\kappa_l + \kappa_{\mathrm{m}} \equiv r^{-1} \sin \psi + (\sin \psi)_r = -u, \tag{8}$$

the left side of which splits the mean curvature of the rotation surface defined by $u(r)$ into a sum of latitudinal (κ_l) and meridional (κ_{m}) sectional curvatures. We note by I ii that $u(r)$ is increasing in $0 < r < R$; thus

$$r^{-1} \sin \psi = -r^{-2}\int_0^r \rho u(\rho)\, \mathrm{d}\rho > -\tfrac{1}{2}u(r) \tag{9}$$

in that interval. Integrating (8) with respect to u and noting that $(\sin \psi)_r = -(\cos \psi)_u$ yields, with use of (9),

$$\cos \psi_R > 1 - \tfrac{1}{4}(u_0^2 - u_R^2),$$

which contains the assertion. We infer now from the general existence theorem, applied at $r = R$, that the solution curve either can be continued upward until it crosses the r axis, or else it tends asymptotically to this axis with increasing r. We may however exclude the latter possibility.

I iv. *If* $u(r) < 0$ *in* $a \leqslant r < R < \infty$, *then* $R < a \exp\{-u_a/a \sin \psi(a)\}$.

† A stronger result of this type is given in Concus & Finn (1975*b*).

Proof. From (5) we find $r\sin\psi > a\sin\psi(a)$ in $a \leqslant r < R$. By I ii, $\sin\psi(a) > 0$. Thus,

$$\mathrm{d}u/\mathrm{d}r = \tan\psi > \sin\psi > ar^{-1}\sin\psi(a),$$

and the result follows on an integration.

We have thus established that if $u_0 \geqslant -2$ the solution curve is in its initial trajectory monotonically increasing and can be continued until it crosses the r axis at a point $r = a_1$. To study the further trajectory, we observe that the curve can be continued at least locally across the axis as a solution of (5), and we compare its inclination at a given height h with the inclination of the initial branch at an equal negative height.

I v. *If the curve can be continued monotonically to a height h above the r axis, then its inclination at this height is smaller than the inclination of the initial branch at the height $-h$, that is,*

$$\left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_h < \left.\frac{\mathrm{d}u}{\mathrm{d}r}\right|_{-h}.$$

Proof. We integrate (8) with respect to u between the height $-h$ and h, obtaining

$$\cos\psi|_h - \cos\psi|_{-h} = \int_{-h}^{h} r^{-1}\sin\psi\,\mathrm{d}u > 0.$$

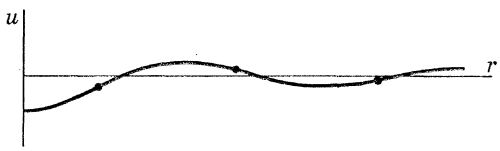

FIGURE 1. The case $u_0 \geqslant -2$: inflexions.

I vi. *Under the conditions of* I iv, *the curve is strictly convex downward when* $u > 0$, *and* $u_{rr} < -u$.

Proof. From (8),

$$(\sin\psi)_r = u_{rr}/(1+u_r^2)^{\frac{3}{2}} = -u - r^{-1}\sin\psi < -u.$$

From I v and I vi we find:

I vii. *The curve can be continued to a maximum height* $h_1 < |u_0|$ *at a point* $r = m_1 > a_1$, *at which point* $\sin\psi(m_1) = 0$.

We now proceed as above, comparing inclinations at corresponding heights until the curve crosses the r axis a second time, then comparing inclinations as in I v, and so on. We obtain the qualitative picture indicated in theorem 1, of a curve oscillating about the r axis with successively decreasing extrema (see figure 1). We note also the additional information, yielded by the method:

I viii. *All inflexions of the curve occur on (monotone) curve segments approaching the r axis, in the sense of increasing s. At any two successive points* α, β, *at which* $|u(\alpha)| = |u(\beta)|$, *there holds*

$$|\mathrm{d}u/\mathrm{d}r|_\alpha > |\mathrm{d}u/\mathrm{d}r|_\beta.$$

To prove the final statement of theorem 1, we note by (7), I ii and I viii that $|u_r(r;u_0)|$ tends uniformly to zero with u_0; thus the function $v(r;u_0) = u_0^{-1}u(r;u_0)$ tends uniformly to the Bessel function $J_0(r)$ as $u_0 \to 0$.

II. Large $|u_0|$; initial arc

If $|u_0|$ is large the above reasoning fails, and so do the results.

THEOREM 2. *If*† $u_0 \leqslant -2\surd 2$, *there exists a value* r_1, *beyond which* $u(r)$ *cannot be continued as a solution of* (5). *As* $r \nearrow r_1$, $\sin\psi \nearrow 1$.

The proof could proceed by a direct study of the equation, as in §1. We obtain more precise results and also develop techniques that will be needed later if we proceed instead via an obvious comparison principle.

II i. *Let* $v^{(1)}(r)$, $v^{(2)}(r)$ *be functions defined in* $a \leqslant r \leqslant b$ *and such that* $(r\sin\psi^{(1)})_r \geqslant (r\sin\psi^{(2)})_r$. *Suppose* $\sin\psi^{(1)}(a) \geqslant \sin\psi^{(2)}(a)$. *Then* $\sin\psi^{(1)}(b) \geqslant \sin\psi^{(2)}(b)$, *and equality holds if and only if* $v^{(1)} \equiv v^{(2)} +$ const. *on* $a \leqslant r \leqslant b$.

The interest in II i lies in the fact that $r^{-1}(r\sin\Psi)_r$ is exactly twice the mean curvature of the rotation surface defined by $u(r)$, and this circumstance facilitates the choice of comparison surfaces. In the present case we choose as initial comparison surface the sphere of constant mean curvature $-\frac{1}{2}u_0$, with centre at the point $(r,u) = (0, u_0 - 2/u_0)$. Thus, if $v(r)$ describes a vertical section of the sphere, there holds $u(0) = v(0)$, $u(r) < v(r)$ in the interval $0 < r \leqslant -2/u_0$ (see figure 2). Using I ii, we find:

II ii. *The solution* $u(r)$ *of* (5, 6) *can be continued at least until* $r = -2/u_0$, *and* $\sin\psi(r) < -\frac{1}{2}u_0 r$.

We also need:

II iii. *A solution* $u(r)$ *of* (5) *admits no inflexions in the region* $ru < -1$.

Proof. From (8) follows $ru + \sin\psi = 0$ at any inflexion.

Thus, ψ must continue to increase until either a vertical point is reached or the curve meets again the hyperbola $ru = -1$. Integration of (8) with respect to u and use of II ii yields

$$1 - \cos\psi > -\tfrac{1}{2}(u^2 - u_0 u).$$

From this we conclude that a vertical slope appears at a value

$$u_1 < \tfrac{1}{2}u_0[1 + (1 - 8u_0^{-2})^{\frac{1}{2}}], \tag{10}$$

which completes the proof of theorem 2.

We may use a similar procedure to estimate the value r_1. We note that if $w(r)$ describes a vertical section of the sphere of constant mean curvature

$$\beta^{-1} = -\tfrac{1}{4}u_0[1 + (1 - 8u_0^{-2})^{\frac{1}{2}}] \tag{11}$$

with centre at $(0, u_0 + \beta)$, then there holds $u(0) = w(0)$, and by II i $u_r(r) > w_r(r)$ on any interval $0 < r \leqslant R$ along which $\beta u \leqslant -2$. This condition is however satisfied at $u = u_1$ by (10), hence on the entire arc $u_0 < u \leqslant u_1$. We conclude $u(r) > w(r)$ until the first vertical occurs at $r = r_1 < \beta$.

We note that at $r = \beta$, where $w_r(r) = \infty$, the circle $w(r)$ intersects the hyperbola $ru = -2$.

From II iii we conclude the initial solution curve is convex in the region $ru < -1$. This property

† This improves the result announced in Concus & Finn (1974).

holds in fact for the entire arc; on the segment of $u(r)$ joining the initial point to the point (r_c, u_c) on the hyperbola $ru = -1$ we obtain from (7, 8, I ii), using the comparison circle $v(r)$,

$$\kappa_m = (\sin\psi)_r = -u - r^{-1}(\sin\psi) > -v + \tfrac{1}{2}u_0 > -v(r_c) + \tfrac{1}{2}u_0 > 0.$$

The last relation holds whenever $u_0 \leqslant -\sqrt{2}$, which is the condition that $v(r)$ and the hyperbola $ru = -1$, $u < 0$ intersect. We conclude also from I ii and the relation

$$\sin\psi = -\tfrac{1}{2}ru + \frac{1}{2r}\int_0^r \rho^2 u_r(\rho)\,d\rho, \tag{12}$$

that $ru > -2$ on the arc considered.

It turns out the sectional curvatures κ_l and κ_m are both monotone decreasing on the initial arc. We have

$$\frac{d}{dr}\kappa_l = \frac{d}{dr}\frac{\sin\psi}{r} = \frac{2}{r^3}\int_0^r \rho u(\rho)\,du - \frac{u}{r} = \frac{1}{r}\left(-2\frac{\sin\psi}{r} - u\right) < 0 \tag{13}$$

by (8, 12). Also, we have from (7), (8) and I ii,

$$\left.\begin{aligned}\frac{d}{dr}\kappa_m &= (\sin\psi)_{rr} = -u_r + \frac{1}{r}\left(2\frac{\sin\psi}{r} + u\right)\\ &< -u_r + \frac{1}{r}(u - u_0) = -\frac{1}{r}\int_0^r \rho u_{rr}(\rho)\,d\rho < 0,\end{aligned}\right\} \tag{14}$$

by the convexity of u.

At the initial point $(0, u_0)$ there holds $\kappa_l = \kappa_m = -\tfrac{1}{2}u_0$. From (8, 12, I ii) follows for $r > 0$ on the initial arc

$$\kappa_m < -\tfrac{1}{2}u < \kappa_l. \tag{15}$$

From (7, 8) we have also

$$\kappa_m = -u - r^{-1}(\sin\psi) > -u + \tfrac{1}{2}u_0. \tag{16}$$

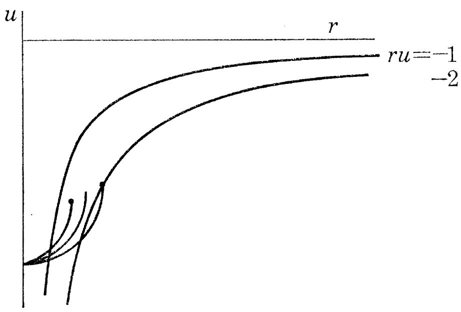

FIGURE 2. Initial comparison surfaces, $u_0 \leqslant -2\sqrt{2}$.

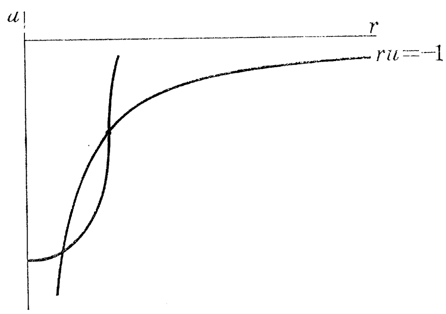

FIGURE 3. The case $u_0 = u_{0c}$.

The inequality (15) implies $\kappa_m < -\tfrac{1}{2}u_0$, which is the meridional curvature of the comparison surface $v(r)$. Comparing the surface $u(r)$ with $v(r)$ at corresponding values of u and applying II ii now yields

$$u_1 > v(-2/r_0) = u_0 - 2/u_0 \tag{17}$$

(see figure 2).

We summarize the above results:

THEOREM 3. *Under the condition of theorem 2, the initial arc of the solution curve, from (r_0, u_0) to (r_1, u_1), is convex, with sectional curvatures κ_m, κ_l decreasing and satisfying $\kappa_m < -\tfrac{1}{2}u < \kappa_l$ in $r_0 < r \leqslant r_1$. There holds*†

$$\left.\begin{aligned}-2/u_0 < r_1 &< -\tfrac{1}{2}u_0[1-(1-8u_0^{-2})^{\frac{1}{2}}],\\ u_0 - 2/u_0 < u_1 &< u_0 - \tfrac{1}{2}u_0[1-(1-8u_0^{-2})^{\frac{1}{2}}].\end{aligned}\right\} \tag{18}$$

† This improves the result announced in Concus & Finn (1974).

For $0 < r \leqslant -2/u_0$ *the arc lies below the comparison circle* $v(r)$ *and has smaller curvature, and for* $u_0 < u \leqslant u_1$ *the arc lies above the comparison circle* $w(r)$ *and has larger curvature* (see figure 2).

Further remark. The hypothesis $u_0 \leqslant -2\sqrt{2}$ of theorem 2 could be sharpened by using the comparison surfaces $v(r)$ and $w(r)$ in (7) and iterating. A direct numerical integration of (7) yields (Concus 1968) $u_0 \approx -2.5678$ as the value for which a vertical first appears. We find immediately:

II iv. *Let* u_{0c} *be the largest value of* u_0 *for which a vertical point appears. If* $u_0 = u_{0c}$ *the vertical occurs at the second intersection of the solution curve with the hyperbola* $ru = -1$, *and is an inflexion point for the solution curve* (see figure 3).

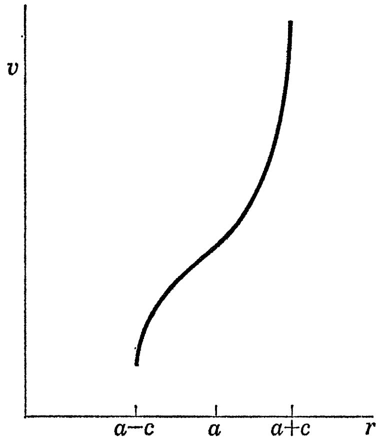

Figure 4. Delaunay surface.

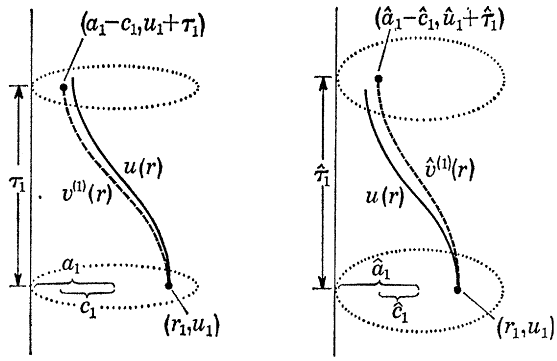

Figure 5. Comparison with roulades.

If $u_0 \leqslant -5$, the upper bounds in (18) can be expressed more simply, yielding

$$\left.\begin{aligned} \frac{-2}{u_0} &< r_1 < -\frac{2}{u_0} - \frac{5}{u_0^3}, \\ u_0 - \frac{2}{u_0} &< u_1 < u_0 - \frac{2}{u_0} - \frac{5}{u_0^3}. \end{aligned}\right\} \tag{19}$$

These bounds could also be improved by iteration, starting with the comparison surfaces $v(r)$ and $w(r)$. We note for reference that the asymptotic series obtained in Concus (1968) by formal perturbation expansion yields, for the normalization used here,

$$\left.\begin{aligned} r_1 &= -\frac{2}{u_0} - \frac{4}{3u_0^3} + O(u_0^{-5}), \\ u_1 &= u_0 - \frac{2}{u_0} - \frac{4 + 8\ln 2}{3u_0^3} + O(u_0^{-5}), \end{aligned}\right\} \tag{20}$$

as $u_0 \to -\infty$.

III. Very large $|u_0|$

If $u_0 \leqslant -2\sqrt{2}$ then $u(r)$ cannot be continued beyond r_1 as a solution of (2). The curve can, however, be continued as a solution of the parametric system (3) as long as r remains different from zero. We study now the behaviour of this family of solutions in terms of the parameter u_0, asymptotically as $u_0 \to -\infty$. We present here and in § IV the first steps in our study of the behaviour of this family of solutions in terms of the parameter u_0, as $u_0 \to -\infty$. The further continuation of the solution is discussed in §§ VI and VII.

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We base the discussion principally on II i; to do so, we introduce as comparison functions the sections of rotation surfaces generated by the roulades of an ellipse. The following result is due to Delaunay (1841):

Let an ellipse of major axis $2a$ and distance $2c$ between focal points, roll rigidly on an axis without slipping. Let $\mathscr{C}$ be the curve swept out by one of the focal points. Then the surface generated by rotating $\mathscr{C}$ about the axis has constant mean curvature $H = (2a)^{-1}$.

We note that $\mathscr{C}$ is periodic with half-period τ satisfying $2a < \tau < \pi a$, and that each half-period can be represented in the interval $a-c \leqslant r \leqslant a+c$ by a single valued function $v(r)$ for which the equation

$$(r \sin\psi)_r/r = 1/a \tag{21}$$

holds, and for which $\sin\psi = 1$ at the two end points (see figure 4).

We proceed step by step:

The procedure of § II shows that an infinite slope first appears at (r_1, u_1), with bounds on (r_1, u_1) given by (18). The system (3) is non-singular at (r_1, u_1), hence the curve can be continued beyond this point as a solution of (3, 4). From (16) we find at (r_1, u_1)

$$\kappa_{\mathrm{m}} > -u_1 + \tfrac{1}{2}u_0 > -\tfrac{1}{2}u_0(1-8u_0^{-2})^{\frac{1}{2}} > 0$$

so the curve turns back towards the u axis, and can be described again (locally) as a solution of (5). We compare it with a roulade $v^{(1)}(r)$ whose mean curvature is $-\frac{1}{2}u_1$ and for which $a_1 + c_1 = r_1$ (see figure 5). Since $v_r^{(1)}(r_1) = -\infty$, II i yields $u_r < v_r^{(1)}$, hence $u(r) > v^{(1)}(r)$ as long as the continuation of both u and $v^{(1)}$ as single valued functions is possible.

The curve $v^{(1)}(r)$ can be continued toward the u axis only until the point $(a_1 - c_1, u_1 + \tau_1)$, with $a_1 - c_1 = -2/u_1 - r_1 > 0$; at this point the slope is again infinite. It follows there is a value $r_2 > -2/u_1 - r_1$ beyond which this branch of the solution curve cannot be continued as a single valued function.

From the geometrical interpretation of τ_1 as the half-circumference of an ellipse with major axis $2a_1 = -2/u_1$ and focal length $c_1 = r_1 - a_1$, one finds that for large $|u_0|$,

$$\tau_1 = -\frac{2}{u_1} + \alpha_1 \frac{\ln|u_1|}{|u_1|^3}, \quad \alpha_1 = -\tfrac{16}{3} + O\left(\frac{1}{\ln|u_1|}\right). \tag{22}$$

Let us estimate r_2 from above. To do so, we compare $u(r)$ with a roulade $\hat{v}_1(r)$, which is determined by the conditions

$$\left.\begin{aligned} \hat{a}_1 &= -1/(u_1 + \hat{\tau}_1), \\ \hat{a}_1 + \hat{c}_1 &= r_1, \\ \hat{\tau}_1 &= \int_0^{\pi} (a^2 - c^2\cos^2\phi)^{\frac{1}{2}}\,\mathrm{d}\phi. \end{aligned}\right\} \tag{23}$$

A formal estimate shows such a roulade exists if $u_1 < -2\pi^{\frac{1}{2}}$.

The conditions (23) are chosen so that the roulade can be placed with its lower vertical point at (r_1, u_1) (see figure 5), and so that in that configuration its mean curvature will be exactly the one determined from the right side of (5) by the upper vertical. Applying II i we obtain $u_r > \hat{v}_r^{(1)}(r)$, $u(r) < \hat{v}^{(1)}(r)$ for all $r < r_1$ for which $u(r) < u_1 + \hat{\tau}_1$. This condition clearly holds for r near r_1; since $\hat{v}^{(1)}(r) < u_1 + \hat{\tau}_1$, we conclude it holds on the entire interval $\hat{a}_1 - \hat{c}_1 < r < r_1$, thus

$$0 > v_r^{(1)}(r) > u_r(r) > \hat{v}_r^{(1)}(r) > -\infty$$

on this interval, and hence the solution can be continued to the left of r_1, at least until the value

$$r_2 < -2/(u_1 + \hat{\tau}_1) - r_1 = \beta_2. \tag{24}$$

For large $|u_0|$ we find

$$\hat{\tau}_1 = -2u_1^{-1} + \hat{\alpha}_1 |u_1|^{-3} \ln |u_1|, \tag{25}$$

with

$$\hat{\alpha}_1 = -\tfrac{40}{3} + O\left(\frac{1}{\ln |u_1|}\right). \tag{26}$$

Thus

$$r_2 < -2u_1^{-1} - \alpha_2 u_1^{-3} - r_1, \tag{27}$$

with

$$\alpha_2 = 4 + O(u_1^{-2} \ln |u_1|). \tag{28}$$

We now proceed, essentially, as in the proof of theorem 2. We note that

$$\sin\psi > \sin\psi^{(1)} = -\tfrac{1}{2}ru_1 + r_1(1 + \tfrac{1}{2}r_1 u_1)/r;$$

thus from (24–28) we find for $r < \beta_2$ that

$$r^{-1} \sin\psi > -\tfrac{3}{50}u_1^{-3} + O(u_1^{-5} \ln |u_1|).$$

We integrate (8) in u from $u(\beta_2)$; using the fact that $\cos\psi < 0$ until a vertical is reached, and that

$$\cos\psi(\beta_2) > \cos\psi^{(1)}(\beta_2) = -\tfrac{1}{5}\sqrt{21} + O(u_1^{-2} \ln |u_1|),$$

we are led to a contradiction unless the curve becomes vertical before u has increased by a value $-16u_1^{-3}$. That is, a vertical must appear at a value

$$u_2 < u_1 + \hat{\tau}_1 - 16u_1^{-3}. \tag{29}$$

The solution curve then turns back from the axis at (r_2, u_2) and initiates a further branch.

We have shown for sufficiently large $|u_0|$.

THEOREM 4. *From (r_1, u_1) the solution curve continues backwards towards the u axis until a second vertical is reached, at a point (r_2, u_2) with*

$$\left.\begin{aligned} -2/u_1 - r_1 < r_2 < -2/(u_1 + \tau_1) - r_1 = \beta_2, \\ v^{(1)}(\beta_2) < u_2 < u_1 + \hat{\tau}_1 - 16u_1^{-3}. \end{aligned}\right\} \tag{30}$$

In the interval $r_2 < r < r_1$ there holds $u_r < v_r^{(1)}, u > v^{(1)}$; in the interval $\beta_2 < r < r_1$ there holds $u_r > \hat{v}_r^{(1)}$, $u < \hat{v}_r^{(1)}$.

We note in particular that the horizontal distance of the second vertical from the axis exceeds that of the first vertical from the hyperbola $ru = -2$.

III i. *There is exactly one inflexion between (r_1, u_1) and (r_2, u_2).*

Proof. Clearly, at least one inflexion appears. Using (8), we find

$$(ru + \sin\psi)_r = (r/\cos\psi - 1/r) \sin\psi < 0$$

on the arc. Hence there is at most one inflexion.

We indicate briefly one further step in the procedure. We construct a roulade $v^{(2)}(r)$ passing through (r_2, u_2), with major axis $2a_2 = -2/u_2$, and a second roulade $\hat{v}^{(2)}(r)$ with a property analogous to that introduced for $\hat{v}^{(1)}(r)$. Then there holds $\hat{v}_r^{(2)} < u_r < v_r^{(2)}$, $\hat{v}^{(2)} < u < v^{(2)}$ in the intervals for which the comparison makes sense, and (as before) still another point (r_3, u_3) is found such that $\sin\psi(r_3) = 1$. The procedure can be continued as long as the values of $|u(r)|$ remain sufficiently large to justify the indicated steps. A detailed description is given in §§ VI and VII.

We find easily:

III ii. *The successive horizontal distances of the vertical points, from the axis and from the hyperbola, increase monotonically.*

III iii. *On each arc segment returning from the hyperbola to the axis there is exactly one inflexion. The same statement holds on the remaining arc segments for sufficiently large* $|u|$.

The following result holds for any $u_0 < 0$.

THEOREM 5. *In the initial region* $u < 0$, *the entire curve is bounded (strictly) between the u axis and the hyperbola* $ru = -2$ (see figure 6). *In this region, the curve can be represented by a single valued function* $r = r(u)$, *with* $|\mathrm{d}r(u)/\mathrm{d}u| < \infty$.

Proof. We note from (8) that at any vertical point not preceded by a horizontal point distinct from $(0, u_0)$ there holds

$$(\sin\psi)_r = -(ru+1)/r,$$

and thus each such point continues to an outgoing arc or returning arc according as $ru > -1$ or $ru < -1$. We integrate (5) on an outgoing arc starting from (r_α, u_α), α even $(r_0 = 0)$, to obtain

$$r\sin\psi = r_\alpha - \int_{r_\alpha}^{r} \rho u(\rho)\,\mathrm{d}\rho,$$

and thus $u_r(r) > 0$ on any such arc along which $u < 0$. We find

$$r\sin\psi - r_\alpha = \tfrac{1}{2}(u_\alpha r_\alpha^2 - ur^2) + \frac{1}{2}\int_{r_\alpha}^{r} \rho^2 u_r(\rho)\,\mathrm{d}\rho > -\tfrac{1}{2}r_\alpha - \tfrac{1}{2}r(ur),$$

from which

$$ur > r_\alpha/r - 2\sin\psi > -2.$$

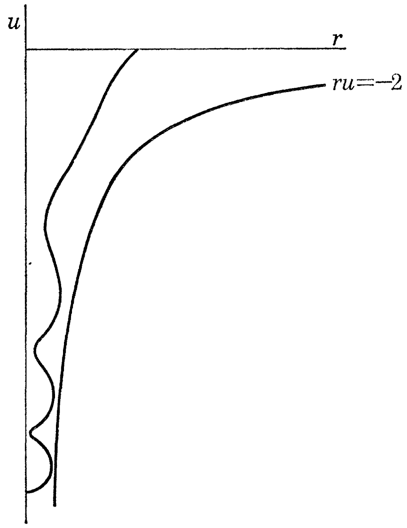

FIGURE 6. The initial region $u < 0$.

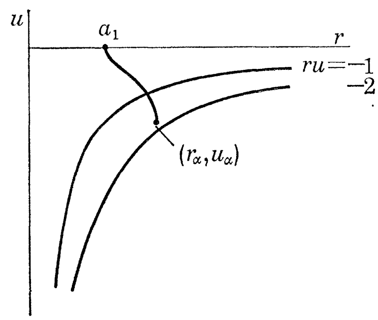

FIGURE 7. Proof of IV i.

On an arc returning from (r_β, u_β) we obtain from (5)

$$r_\beta - r\sin\psi = \tfrac{1}{2}(ur^2 - u_\beta r_\beta^2) + \frac{1}{2}\int_{r}^{r_\beta} \rho^2 u_r(\rho)\,\mathrm{d}\rho,$$

and since $u_\beta r_\beta > -2$,

$$\sin\psi > -\tfrac{1}{2}ur - \tfrac{1}{2}r^{-1}\int_{r}^{r_\beta} \rho^2 u_r(\rho)\,\mathrm{d}\rho,$$

from which we conclude easily $u_r(\rho) < 0$ in the region $u < 0$. There follows immediately $r > r_\alpha > 0$ along such an arc.

IV. Global behaviour

The discussion thus far shows that the solution curve can be continued upward without self-intersections until it crosses the r axis. For by I iii an outward branch must either achieve a vertical or cross that axis, and the comparison method of § II yields readily that a returning branch has the same property. There are no horizontal points, by theorem 5.

We show here that a returning branch cannot cross the r axis. Precisely:

IV i. *Let $r = a_1$ be the first point at which the solution curve meets the r axis. Then* $0 < u_r(a_1) < \infty$.

Suppose $u_r(a_1) < 0$, or equivalently, $\cos\psi_1 < 0$. The curve could then be continued backward into the negative u-plane until a first vertical (r_α, u_α) (see figure 7), at which, by theorem 5,

$$r_\alpha u_\alpha > -2. \tag{31}$$

We integrate (8) with respect to u, from u_α to 0, obtaining

$$\int_{u_\alpha}^{0} r^{-1}\sin\psi\, du = \cos\psi_1 + \tfrac{1}{2}u_\alpha^2. \tag{32}$$

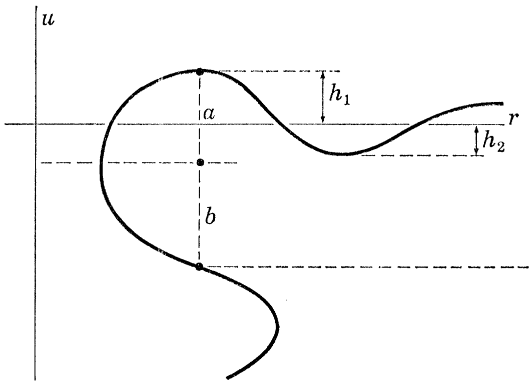

Figure 8. Proof of theorem 6.

To evaluate the left side of (32) we integrate (5) in r between r and r_α:

$$r_\alpha - r\sin\psi = -\int_r^{r_\alpha} \rho u(\rho)\, d\rho < \tfrac{1}{2}(r^2 - r_\alpha^2)\, u_\alpha$$
$$< \tfrac{1}{2}r^2 u_\alpha + r_\alpha,$$

by (31). Thus

$$r^{-1}\sin\alpha > -\tfrac{1}{2}u_\alpha, \tag{33}$$

on the entire arc. Placing (33) into (32) yields $\cos\psi_1 > 0$, contradicting the assumption.

Now observe from (8) that at the crossing point a_1, the meridional curvature is negative; thus, if $\cos\psi_1 = 0$ there would again be a backward branch from a_1 into the negative u-plane, and we obtain a contradiction as above.

From IV i one sees immediately that the proof of theorem 1 applies without change to the region $r \geq a_1$, in the sense that the solution curve continues from the point $(a_1, 0)$ as indicated in figure 1. We show now the curve does not intersect the initial branch in the region $u < 0$.

IV ii. *Let u_α, u_β be two successive points on the solution curve such that $r_\alpha = r_\beta$, with an intervening vertical at (r_γ, u_γ). If $r_\gamma < r_\alpha$, then $\sin\psi_\beta < \sin\psi_\alpha$; if $r_\gamma > r_\alpha$, then $\sin\psi_\beta > \sin\psi_\alpha$.*

Proof. Suppose $r_\gamma < r_\alpha$. From (5) we find

$$r_\beta \sin\psi_\beta - r_\gamma = -\int_{r_\gamma}^{r_\beta} \rho u(\rho)\, \mathrm{d}\rho,$$

$$r_\alpha \sin\psi_\alpha - r_\gamma = -\int_{r_\gamma}^{r_\alpha} \rho u(\rho)\, \mathrm{d}\rho,$$

and thus since $r_\alpha = r_\beta$, $$r_\alpha(\sin\psi_\beta - \sin\psi_\alpha) = \int_{r_\gamma}^{r_\alpha} \rho(u^- - u^+)\, \mathrm{d}\rho < 0;$$

u^- and u^+ denote values of u on the lower and upper branches. The case $r_\gamma > r_\alpha$ follows similarly.

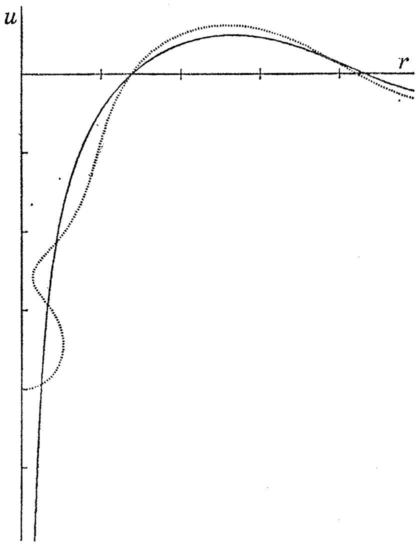

FIGURE 9. $u_0 = -4$; singular solution.

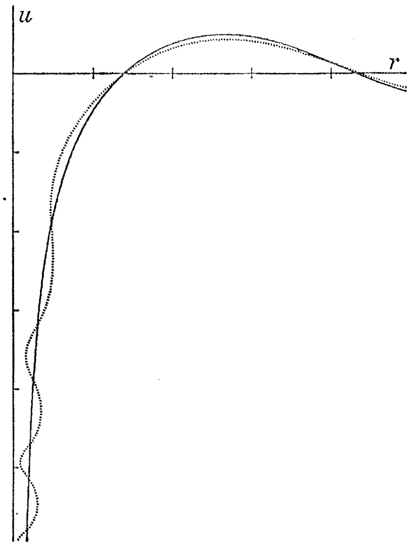

FIGURE 10. $u_0 = -8$; singular solution.

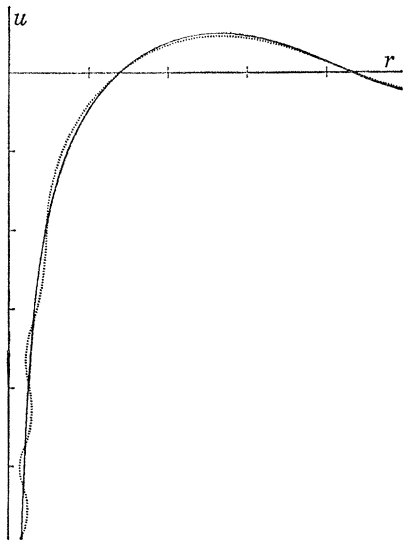

FIGURE 11. $u_0 = -16$; singular solution.

From IV ii it follows that $a < b$ in figure 8, and thus, $h_1 < b$. But $h_j < h_1$, any $j > 1$, by I viii, hence $h_j < b$, all $j > 1$, and thus intersections are excluded.

Combining these results with theorems 1 and 5 we obtain:

THEOREM 6. *The solution of the parametric system* (3, 4) *defined by the data* u_0 *can be continued indefinitely without limit sets or double points. It has the form indicated in figures* 1, 6, 9, 10 and 11.

V. MAXIMUM DIAMETER

We define the *diameter* of a (symmetric) liquid drop as the largest diameter of all circular sections $u = u_j$, at which the bounding surface is vertical.

From theorems 1, 5 and 6 we see that each drop has a well defined diameter. It is less obvious that there is a universal upper bound for the diameters of all possible drops, independent of u_0.

THEOREM 7. *Let* $\delta \approx 2.473$ *be the unique positive root of the equation*

$$r^3 - 3^{\frac{3}{2}} r - 3^{\frac{3}{4}} = 0. \tag{34}$$

Then 2δ *exceeds the diameter of any solution of* (3, 4).

We base the proof on a lemma, which also has an independent interest.

V i. *Let* $u(r)$ *represent a solution curve passing through* (a, u_a) *with* $-1 \leqslant a u_a < 0$, *and such that*

$$a \sin\psi_a \geqslant \tfrac{1}{2} a. \tag{35}$$

Suppose $u(r) < 0$ in $a \leqslant r < R$. Then $\sin\psi > 0$ on this arc segment. If the curve meets the hyperbola $ru = -1$ in a point (c, u_c) with $a < c < R$, then $c < 3^{\frac{1}{4}}$, and $\sin\psi_c > \frac{1}{2}$.

Proof. We integrate (5) between α and r, obtaining

$$r\sin\psi - \alpha\sin\psi_\alpha = \tfrac{1}{2}(\alpha^2 u_\alpha - r^2 u(r)) + \frac{1}{2}\int_\alpha^r \rho^2 u_r(\rho)\,\mathrm{d}\rho, \tag{36}$$

from which, if $\alpha = a$,

$$r\sin\psi \geqslant -\tfrac{1}{2}r^2 u(r) + \frac{1}{2}\int_a^r \rho^2 u_r(\rho)\,\mathrm{d}\rho. \tag{37}$$

For r sufficiently near a, there holds $\sin\psi > 0$. Thus, if $\sin\psi$ were to vanish at any points interior to $a \leqslant r < R$, there would be a minimum $r = r_\gamma > a$ at which this occurs. But (37) would then imply

$$0 = r_\gamma \sin\psi_\gamma > \frac{1}{2}\int_a^{r_\gamma} \rho^2 u_r(\rho)\,\mathrm{d}\rho > 0,$$

a contradiction. Thus, $\sin\psi > 0$ on $a \leqslant r \leqslant c$, and hence $u_r(\rho) > 0$ on this interval. Setting now $r = c$ in (37) yields

$$\sin\psi_c > -\tfrac{1}{2}cu(c) = \tfrac{1}{2}. \tag{38}$$

Finally, we note that at $r = c$ the inclination of the solution curve cannot exceed that of the hyperbola. Thus, $\sin\psi_c < (1+c^4)^{-\frac{1}{2}}$, and $c^4 < 3$ follows from (38).

We proceed to prove theorem 7. For any given u_0, the maximum width is attained at a point (r_{2j+1}, u_{2j+1}) with $-1 \geqslant r_{2j+1}u_{2j+1} > -2$, $j \geqslant 0$ (see § III). At the preceding point (r_{2j}, u_{2j}) there holds either $r_{2j} = 0$ (if $j = 0$), or else $\sin\psi_{2j} = 1$. In either event, (35) holds with $a = r_{2j}$. Also $-1 < r_{2j}u_{2j} < 0$, and thus the curve crosses the hyperbola $ru = -1$ at a point (c, u_c), $r_{2j} < c < r_{2j+1}$. Setting $\alpha = c$, $r = r_{2j+1}$ in (36) and applying V i yields, by use of II iii,

$$r_{2j+1}^3 - 3^{\frac{3}{2}} r_{2j+1} - 3^{\frac{3}{4}} < 0. \tag{39}$$

The (single) positive solution of (34) exceeds any solution of (39). Since j is arbitrary, we conclude 2δ exceeds the diameter of any drop.

VI. Generic estimates for large $|u|$

The solutions discussed in this paper are apparently related to a singular solution $U(r)$ of (2), whose existence we have already proved (Concus & Finn 1975*a*). The function $U(r)$ is defined in a deleted neighbourhood of $r = 0$, and there holds asymptotically $U(r) \simeq -r^{-1}$, as $r \to 0$. We have conjectured that in any interval $0 < a \leqslant r \leqslant b < \infty$, the solutions of (3, 4) admit a single valued representation $u(r; u_0)$ and converge uniformly to $U(r)$, as $u_0 \to -\infty$. Figures 9–11 show the results of calculations supporting the conjecture.

In this section and in the following one we develop asymptotic properties of the solutions, which again support the conjecture, although they do not yet settle it completely. The properties are described in general terms below and in detail in theorems 8 and 9, and seem of independent interest.

The crucial new step in the present discussion consists in a more precise use of the Delaunay comparison surfaces as a device to control the behaviour of the solutions of

$$(r\sin\psi)_r = -ru. \tag{5}$$

In § III we used bounds on these surfaces for estimation of integrals of the right side of (5); we now propose to introduce the Delaunay profiles themselves into these integrals. It turns out the results can be expressed succinctly in terms of elliptic integrals, leading to an improvement in an order of magnitude of the estimates of § III. We are led after some manipulation to the underlying recurrence relations (107), (136) and (137) for the displacement of successive 'vertical points' from the hyperbola $ru = -1$, which show that, initially, the solution curve becomes closer to the hyperbola with each successive loop. Integration of these relations shows that the solution curve contracts toward the hyperbola at least until a height of order $|u_0|^{\frac{7}{9}}$, after which it remains confined within a strip whose width has order $|u_0|^{-1}$, until a height of order $|u_0|^{(2\alpha+1)/9}$, for any $\alpha > \frac{23}{9}$. Thus, the solution curve converges asymptotically to the hyperbola, uniformly in $|u| > |u_0|^{(2\alpha+1)/9}$.

For all sufficiently large $|u|$, we show the solution curve is confined to a strip about $ru = -1$, whose width has order $|u|^{-\frac{9}{7}}$, uniformly in u_0 as $|u_0| \to \infty$.

It will be convenient to use here a somewhat different notation than was employed in § III, which seems better adapted to the description of generic 'loops'. We use also the symbols A, B, to denote quantities independent of the other terms within a relation, but whose values may however change within a context. Thus, from $y < A(1+x^2)^{\frac{1}{2}}$ we may conclude $y < A|x|$ for large $|x|$. The symbol $\sim$ is used to indicate a relation in which terms of (relatively) small magnitude are neglected. The notation $A \gg B$ means $A-B$ is (sufficiently) large, depending on the context, but independent of u_0 for $|u_0|$ sufficiently large.

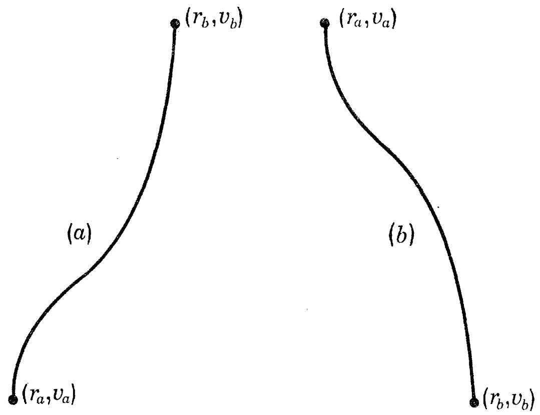

FIGURE 12. Delaunay arcs. (*a*) Outgoing, (*b*) returning.

We start with general estimates on Delaunay arcs $v(r)$, which are solutions of

$$(r \sin \psi)_r = 2rH, \quad H > 0, \tag{40}$$

vertical at (r_a, v_a), (r_b, v_b), $r_a < r_b$ (see figure 12). We note

$$H = 1/(r_a + r_b), \tag{41}$$

and an inflexion appears at

$$r_i = (r_a r_b)^{\frac{1}{2}}. \tag{42}$$

We distinguish two cases:

CASE (*a*) $\psi \leqslant \frac{1}{2}\pi$. Solving for $r(\psi)$, we find

$$r = \frac{\sin \psi \pm (k^2 - \cos^2 \psi)^{\frac{1}{2}}}{2H}, \quad k = \frac{r_b - r_a}{r_b + r_a}, \tag{43}$$

where the upper (lower) sign is to be chosen, according as $r > (<)\, r_i$.

Setting $\cos\psi = k\sin\varphi$, and using $u_r(r) = \tan\psi$, we integrate (43) to obtain†

$$v_b - v_a = \frac{1}{H}\int_0^{\frac{1}{2}\pi}(1-k^2\sin^2\varphi)^{\frac{1}{2}}\,\mathrm{d}\varphi = \frac{1}{H}E(k), \tag{44}$$

where $E(k)$ is the elliptic integral of the second kind, of modulus k.

If $v_i < v < v_b$, then
$$v - v_a = E(k)/H - [k\sin\varphi + E(\varphi, k)]/2H, \tag{45}$$

where
$$E(\varphi, k) = \int_0^{\varphi}(1-k^2\sin^2\tau)^{\frac{1}{2}}\,\mathrm{d}\tau$$

is the incomplete elliptic integral of the second kind.

If $v_a < v < v_i$, then
$$v - v_a = [-k\sin\varphi + E(\varphi, k)]/2H. \tag{46}$$

At the inflexion (r_i, v_i),
$$v_i = [-k + E(k)]/(2H) + v_a. \tag{47}$$

Case (b) $\psi \geqslant \frac{1}{2}\pi$. The discussion is unchanged, except in this case $-\frac{1}{2}\pi \leqslant \varphi \leqslant 0$. We find now

$$v_a - v_b = E(k)/H. \tag{48}$$

If $v_b \leqslant v \leqslant v_i$, then
$$v - v_b = -[k\sin\varphi + E(\varphi, k)]/2H. \tag{49}$$

If $v_i \leqslant v \leqslant v_a$, then
$$v - v_b = E(k)/H - [k\sin\varphi - E(\varphi, k)]/2H. \tag{50}$$

We have, in this case,
$$v_i = [k + E(k)]/(2H) + v_b. \tag{51}$$

We shall need to evaluate integrals of Delaunay arcs, of the form

$$\begin{aligned}\mathscr{I}_{ab} &= -\int_{r_a}^{r_b}\rho v(\rho)\,\mathrm{d}\rho \\ &= -\tfrac{1}{2}\rho^2 v|_{r_a}^{r_b} + \frac{1}{2}\int_{r_a}^{r_b}\rho^2\frac{\mathrm{d}v}{\mathrm{d}\rho}\,\mathrm{d}\rho \\ &= -\tfrac{1}{2}\rho^2 v|_{r_a}^{r_b} + \frac{1}{2}\left(\int_0^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{0}\right)\rho^2\frac{\mathrm{d}v}{\mathrm{d}\varphi}\mathrm{d}\varphi,\end{aligned} \tag{52}$$

for the case $\psi \leqslant \frac{1}{2}\pi$; the last two integrals refer to the portions of the curve preceding and following the inflexion. For $\psi \geqslant \frac{1}{2}\pi$, $\varphi \leqslant 0$ and the limits in the last two integrals become $-\frac{1}{2}\pi$.

Case (a). $\psi \leqslant \frac{1}{2}\pi$. Taking $r(\varphi)$, $v(\varphi)$ from (43), (45) and (46), and setting $\Delta(\varphi, k) = (1-k^2\sin^2\varphi)^{\frac{1}{2}}$, we find, according as $r \gtrless r_i$,

$$r^2\frac{\mathrm{d}v}{\mathrm{d}\varphi} = \frac{1}{8H^3}\{\mp(1-k^2)\,\Delta \mp 4k^2\Delta\cos^2\varphi - 3k\Delta^2\cos\varphi - k^3\cos^3\varphi\}. \tag{53}$$

After taking account of some cancellation, we obtain

$$\begin{aligned}\mathscr{I}_{ab} &= -\tfrac{1}{2}[r_b^2 v_b - r_a^2 v_a] + \frac{1}{8H^3}\int_0^{\frac{1}{2}\pi}[(1-k^2)\,\Delta + 4k^2\Delta\cos^2\varphi]\,\mathrm{d}\varphi \\ &= -\tfrac{1}{2}[r_b^2 v_b - r_a^2 v_a] + S(k)/24H^3,\end{aligned} \tag{54}$$

with
$$S(k) = 8E(k) - (1-k^2)\,(E(k) + 4K(k)), \tag{55}$$

where
$$K(k) = \int_0^{\frac{1}{2}\pi}(1-k^2\sin^2\varphi)^{-\frac{1}{2}}\,\mathrm{d}\varphi$$

is the complete elliptic integral of the first kind, of modulus k.

† We note for reference the alternative representation $v_b - v_a = r_a K(\tilde{k}) + r_b E(\tilde{k})$, where K and E are complete elliptic integrals of first and second kind, and $\tilde{k} = (r_b^2 - r_a^2)^{\frac{1}{2}}/r_b$. Similarly, (45) takes the form $v - v_a = r_a F(\varphi, \tilde{k}) + r_b[E(\tilde{k}) - E(\Phi, \tilde{k})]$, where F is the incomplete integral of the first kind, and $r(1-\tilde{k}^2\sin^2\varphi)^{\frac{1}{2}} = r_a$, $r_b(1-\tilde{k}^2\sin^2\Phi)^{\frac{1}{2}} = r$. In this form of the representation there is no need to distinguish the inflexion; however, the formulae become technically more complicated in other respects.

CASE (*b*). $\psi \geqslant \frac{1}{2}\pi$. In this case, we find by an analogous discussion

$$\mathscr{I}_{ab} = -\tfrac{1}{2}[r_b^2 v_b - r_a^2 v_a] - S(k)/24H^3. \tag{56}$$

We indicate in a particular configuration how the above expressions can be used to estimate the solution $u(r)$ of (5). We consider an arc $u(r)$ that is vertical at (r_α, u_α) and at (r_β, u_β), $u_\alpha < u_\beta$ (figure 13). We compare this arc with a Delaunay arc $v(r)$, with curvature $H = -\frac{1}{2}u_\alpha$, and vertical at $(r_a, v_a) = (r_\alpha, u_\alpha)$. The second vertical then appears at (r_b, v_b), determined by

$$-1/u_\alpha = \tfrac{1}{2}(r_a + r_b), \tag{57}$$

and by

$$v_b - u_\alpha = -2E(k)/u_\alpha \tag{58}$$

with

$$k = \frac{r_b - r_a}{r_b + r_a} = 1 + r_\alpha u_\alpha. \tag{59}$$

The general comparison principle II i applies in the interval $r_\alpha \leqslant r \leqslant r_b$, and yields $u_r(r) < v_r(r)$, $u(r) < v(r)$ in this interval. A consequence is that

$$r_b < r_\beta, \quad u^* = u(r_b) < v_b. \tag{60}$$

We extend $v(r)$ to the interval (r_α, r_β) by defining $v(r) \equiv u_\beta, r > r_b$. From the equation (5) we now find

$$\begin{aligned} r_\beta - r_\alpha &= -\int_{r_\alpha}^{r_\beta} \rho u(\rho)\,d\rho > -\int_{r_\alpha}^{r_b} \rho v(\rho)\,d\rho - \int_{r_b}^{r_\beta} \rho u_\beta\,d\rho \\ &> -\tfrac{1}{2}[r_b^2 v_b - r_\alpha^2 u_\alpha] + S(k)/24H^3 - \tfrac{1}{2}u_\beta(r_\beta^2 - r_b^2), \end{aligned} \tag{61}$$

which we rewrite in the form

$$\tfrac{1}{2}v_b r_\beta^2 + r_\beta > \tfrac{1}{2}u_\alpha r_\alpha^2 + r_\alpha + S(k)/24H^3 + \tfrac{1}{2}(r_b^2 - r_\beta^2)\,\epsilon_\beta, \tag{62}$$

with $\epsilon_\beta = u_\beta - v_b$.

We can obtain a similar estimate in the reverse direction by introducing a Delaunay surface $\hat{v}(r)$, vertical at (r_α, u_α) and at $(\hat{r}_b, \hat{v}_b)$, and with mean curvature $\hat{H} = -\frac{1}{2}u_\beta$. The comparison principle now yields

$$r_b < r_\beta < \hat{r}_b; \quad \hat{v}(r) < u(r), \quad r_\alpha \leqslant r \leqslant r_\beta; \quad \hat{v}(r_b) < u^* < v_b \tag{63}$$

(see figure 13). Integrating (5) we obtain

$$\begin{aligned} r_\beta - r_\alpha &= -\int_{r_\alpha}^{r_\beta} \rho u(\rho)\,d\rho < -\int_{r_\alpha}^{r_\beta} \rho \hat{v}(\rho)\,d\rho = -\int_{r_\alpha}^{\hat{r}_b} \rho \hat{v}(\rho)\,d\rho + \int_{r_\beta}^{\hat{r}_b} \rho \hat{v}(\rho)\,d\rho \\ &< -\tfrac{1}{2}[\hat{r}_b^2 \hat{v}_b - r_\alpha^2 u_\alpha] + 1S(\hat{k})/24\hat{H}^3 + \tfrac{1}{2}\hat{v}_b(\hat{r}_b^2 - r_\beta^2), \end{aligned} \tag{64}$$

by (56, 63). We rewrite (64) in the form

$$(\tfrac{1}{2}r_b^2 \hat{v}_b + r_\beta) < (\tfrac{1}{2}r_\alpha^2 u_\alpha + r_\alpha) + S(\hat{k})/24\hat{H}^3. \tag{65}$$

In order to extract useful information from (62, 65), we need conditions under which a second vertical will appear, and an estimate for ϵ_β and the consequent estimates on $\hat{H}$, $\hat{k}$; we proceed to obtain them.

We consider, for the case $\psi \leqslant \frac{1}{2}\pi$, the generic configuration indicated in figure 13. Setting, as before, $u^* = u(r_b)$, $\psi^* = \psi(r_b)$, we find from (5)

$$r_b \sin\psi^* - r_\alpha = -\int_{r_\alpha}^{r_b} \rho u(\rho)\,d\rho > -\tfrac{1}{2}v_b(r_b^2 - r_\alpha^2). \tag{66}$$

For the upper Delaunay surface $v(r)$ we have from (40, 41, 57)

$$r_b - r_\alpha = -\tfrac{1}{2}u_\alpha(r_b^2 - r_\alpha^2). \tag{67}$$

Combining these relations, we obtain

$$r_b(1-\sin\psi^*) < \tfrac{1}{2}(v_b - u_\alpha)\,(r_b + r_\alpha)\,(r_b - r_\alpha) \tag{68}$$

so that, by (43), (44) and (57),

$$1-\sin\psi^* < 4E(k)\,\frac{k}{1+k}\,\frac{1}{u_\alpha^2}, \tag{69}$$

from which
$$\cos\psi^* < \frac{2\sqrt{2}\,[E(k)]^{\frac{1}{2}}}{-u_\alpha}\left(\frac{k}{1+k}\right)^{\frac{1}{2}}. \tag{70}$$

We now observe $(\sin\psi)_r = -(\cos\psi)_u$ and write (5) in the form

$$r^{-1}\sin\psi - (\cos\psi)_u = -u. \tag{71}$$

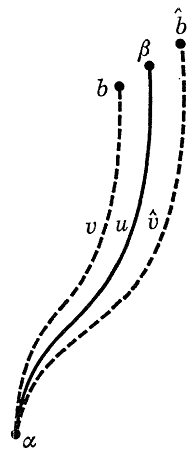

FIGURE 13. Comparison with Delaunay arcs; outgoing case.

For all $r > r_b$ for which the solution can be continued in the form $u = u(r)$, we conclude

$$-(\cos\psi)_u > -u - 1/r_b = -u + u_\alpha/(1+k). \tag{72}$$

Integrating in u between the values u^* and u, observing $\cos\psi > 0$ and using (70), we find

$$\tfrac{1}{2}(\delta u)^2 + \left(u^* - \frac{u_\alpha}{1+k}\right)\delta u - \frac{2\sqrt{2}\,[E(k)]^{\frac{1}{2}}}{u_\alpha}\left(\frac{k}{1+k}\right)^{\frac{1}{2}} > 0, \tag{73}$$

where we have set $\delta u = u - u^*$, on the arc considered. We have also

$$u^* < v_b = u_\alpha - 2E(k)/u_\alpha$$

by (58) and (60), and thus

$$\tfrac{1}{2}(\delta u)^2 + \left(\frac{k}{1+k}u_\alpha - \frac{2E(k)}{u_\alpha}\right)\delta u - \frac{2\sqrt{2}[E(k)]^{\frac{1}{2}}}{u_\alpha}\left(\frac{k}{1+k}\right)^{\frac{1}{2}} > 0 \tag{74}$$

on any continuation of the solution arc to values $u \geqslant u^*$. We conclude that *a second vertical must appear, in every situation for which*

$$ku_\alpha^2 \geqslant 1. \tag{75}$$

Under this condition we obtain from (74) the simple expression for $\delta^* u = \max_{\psi \leqslant \frac{1}{2}\pi} (u - u^*) = u_\beta - u^*$,

$$\delta^* u \lesssim \frac{2\sqrt{2}\,[E(k)]^{\frac{1}{2}}\,[k(1+k)]^{\frac{1}{2}}}{k u_\alpha^2 - 2E(k)\,(1+k)} < \frac{A}{\sqrt{k}\,u_\alpha^2} < \frac{\epsilon}{|u_\alpha|}, \tag{76}$$

which limits the height change between the successive verticals. Here $\epsilon > 0$ is arbitrarily small, for large $k u_\alpha^2$.

We note the condition (76) ensures that the second vertical (r_β, u_β) lies to the right of the hyperbola $ru = -1$, that is, $r_\beta u_\beta < -1$. We show that under this condition, *the hyperbola is crossed exactly once between* (r_α, u_α) *and* (r_β, u_β). To see this, we first observe that the comparison function $v(r)$ has exactly one inflexion, which must appear in the initial interval determined by $rv > -1$. Also the vertical distance from (r_b, v_b) to the hyperbola $ru = -1$ is

$$d_b = -\frac{1}{r_b} - v_b = -\frac{k}{1+k} u_\alpha + \frac{2E(k)}{u_\alpha},$$

which is positive if $k u_\alpha^2 \gg 1$. Thus, $r_b v_b < -1$, and it follows that $v(r)$ meets the hyperbola exactly once. Since by II i, $u_r(r) < v_r(r)$, $u(r)$ meets the hyperbola exactly once in the interval $[r_\alpha, r_b]$. We now observe

$$d_b - \delta^* u > -\frac{k}{1+k} u_\alpha + \frac{2E(k)}{u_\alpha} - \frac{A}{\sqrt{k u_\alpha^2}}, \tag{77}$$

by (76). The condition $k u_\alpha^2 > B$ implies

$$d_b - \delta^* u > \frac{1}{|u_\alpha|} \left(\frac{B}{1+k} - 2E(k) - \frac{A}{\sqrt{B}} \right), \tag{78}$$

which is positive for large B. Thus, $u(r)$ cannot cross the hyperbola in the interval $[r_b, r_\beta]$, which completes the proof.

The result (76) permits us to estimate the error terms in (62) and (65). We find, using (58), (60) and (76),

$$\begin{aligned} 0 < \hat{H} - H &= \tfrac{1}{2}(u_\beta - u_\alpha) = \tfrac{1}{2}(u^* - u_\alpha) + \tfrac{1}{2}\delta^* u < \tfrac{1}{2}(v_\beta - u_\alpha) + \tfrac{1}{2}\delta^* u \\ &= -\frac{E(k)}{u_\alpha} + \tfrac{1}{2}\delta^* u < -\frac{E(k)}{u_\alpha} + \frac{A}{\sqrt{k u_\alpha^2}} \end{aligned} \tag{79}$$

for large ku^2. Similarly, by (57), (58), (60) and (76),

$$\begin{aligned} 0 < r_\beta - r_b < \hat{r}_b - r_b &= 2(u_\beta - u_\alpha)/u_\beta u_\alpha \\ &= -4E(k)/u_\alpha^3 + \delta^* u/u_\alpha^2 + O(|u_\alpha|^{-5}) \end{aligned} \tag{80}$$

uniformly in k. It follows that

$$0 < r_\beta^2 - r_b^2 < A|u_\alpha|^{-4}, \tag{81}$$

again uniformly in k.

We have $k = 1 + r_\alpha u_\alpha$, $\hat{k} = 1 + r_\alpha u_\beta$, so that, by (59) and (76),

$$\begin{aligned} 0 < \hat{k} - k = r_\alpha(u_\beta - u_\alpha) &< r_\alpha(\delta^* u - 2E(k)/u_\alpha) \\ &< (1-k)\,(2E(k)/u_\alpha^2 + \delta^* u/|u_\alpha|) \\ &< (1-k)\,(2E(k) + \epsilon)\,u_\alpha^{-2}, \end{aligned} \tag{82}$$

for large ku_α^2. A formal calculation, using the asymptotic estimates for E and K for $k \approx 1$ (*cf.* Jahnke & Emde (1945), ch V), now yields

$$\left.\begin{aligned} E(\hat{k}) - E(k) &= O(u_\alpha^{-2}) \\ S(\hat{k}) - S(k) &= O(u_\alpha^{-2}) \end{aligned}\right\} \tag{83}$$

uniformly in k. The singularity of K near $k = 1$ is here cancelled by the factor $(1-k)$ in (82).

We note next

$$-(v_b - \hat{v}(r_b)) < \epsilon_\beta < \delta^* u + u^* - v_b < \delta^* u < A u_\alpha^{-2}/\sqrt{k}. \tag{84}$$

We estimate the left side of (84) by using the explicit representation (45) for the surface $\hat{v}(r)$. This representation will apply, as $\hat{v}_i < \hat{v}(r_b)$ for $\hat{k}u_\alpha^2 \gg 1$. In the present case we find

$$v_b - \hat{v}(r_b) = -\frac{2E(k)}{u_\alpha} + \frac{2E(\hat{k})}{u_\alpha - (2E(\hat{k})/u_\beta)} + \frac{1}{u_\alpha - (2E(\hat{k})/u_\beta)}\left[\hat{k}\sin\hat{\varphi}_b + \int_0^{\hat{\varphi}_b} (1 - \hat{k}^2 \sin^2\varphi)^{\frac{1}{2}}\, d\varphi\right]. \tag{85}$$

From the definition of v, $\hat{v}$ we find

$$r_b - r_\alpha = -u_\alpha \int_{r_\alpha}^{r_b} \rho\, d\rho, \tag{86}$$

$$r_b \sin\hat{\psi}(r_b) - r_\alpha = -u_\beta \int_{r_\alpha}^{r_b} \rho\, d\rho. \tag{87}$$

Thus

$$1 - \sin\hat{\psi}(r_b) = \tfrac{1}{2}(u_\beta - u_\alpha)\,(r_b^2 - r_\alpha^2)/r_b, \tag{88}$$

and from $1 - \sin^2\hat{\psi} = \hat{k}^2\sin^2\hat{\varphi}$ there follows, by using (59), (67) and (79),

$$\hat{k}^2 \sin^2\hat{\varphi} < A\frac{k}{1+k} u_\alpha^{-2}. \tag{89}$$

We place this result in (85) and use (79) to obtain

$$0 < v_b - \hat{v}(r_b) < A u_\alpha^{-2}/\sqrt{k}, \tag{90}$$

uniformly in k.

We are now in position to put (62, 65) into more effective forms. We write first, from (62), with $H = -\frac{1}{2}u_\alpha$,

$$\tfrac{1}{2}u_\beta\left(r_\beta + \frac{1}{u_\beta}\right)^2 - \frac{1}{2u_\beta} > \tfrac{1}{2}u_\alpha\left(r_\alpha + \frac{1}{u_\alpha}\right)^2 - \frac{1}{2u_\alpha} - \frac{1}{3u_\alpha^3}S(k) + \tfrac{1}{2}r_b^2\,(u_\beta - v_b), \tag{91}$$

from which

$$\tfrac{1}{2}u_\alpha\left\{\left(r_\beta + \frac{1}{u_\beta}\right)^2 - \left(r_\alpha + \frac{1}{u_\alpha}\right)^2\right\} > \frac{1}{2}\left(\frac{1}{u_\beta} - \frac{1}{u_\alpha}\right) - \frac{1}{3u_\alpha^3}S(k) + \frac{1}{2}\frac{u_\alpha - u_\beta}{u_\beta^2}(r_\beta u_\beta + 1)^2 + \tfrac{1}{2}r_b^2(u_\beta - v_b). \tag{92}$$

We have

$$\begin{aligned} u_\beta - u_\alpha &= u_\beta - v_b + v_b - u_\alpha \\ &= \delta^* u + u^* - v_b + v_b - u_\alpha \\ &= -2E(k)/u_\alpha + \delta^* u + u^* - v_b, \end{aligned} \tag{93}$$

which implies, by (76) and (90), that

$$-\frac{A}{\sqrt{k}\,u_\alpha^2} - \frac{2E(k)}{u_\alpha} < u_\beta - u_\alpha < -\frac{2E(k)}{u_\alpha} + \frac{A}{\sqrt{k}\,u_\alpha^2}. \tag{94}$$

The same calculation yields

$$|u_\beta - v_b| < A u_\alpha^{-2}/\sqrt{k}. \tag{95}$$

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We have also, by (80) and (76),

$$0 < r_\beta - r_b < \hat{r}_b - r_b = -\frac{4E(k)}{u_\alpha^3} + \frac{A}{\sqrt{k}\,u_\alpha^4}, \tag{96}$$

with $|A|$ bounded uniformly in k, u_α for large $|u_\alpha|$, from which we derive

$$\begin{aligned} 1 + r_\beta u_\beta &= 1 + r_b v_b + (r_\beta u_\beta - r_b v_b) \\ &= -k - 2E(k)\, r_b/u_\alpha + r_\beta(u_\beta - v_b) + v_b(r_\beta - r_b), \end{aligned} \tag{97}$$

so that the above estimates yield $$|(1+r_\beta u_\beta) + k| < A u_\alpha^{-2} \tag{98}$$
uniformly in k.

Returning to (92), we may now write

$$\begin{aligned} \tfrac{1}{2}u_\alpha \left\{\left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right)\right\}\left\{(r_\beta - r_\alpha) + \frac{1}{u_\beta} - \frac{1}{u_\alpha}\right\} &> \frac{(1+k^2)\,E(k)}{u_\alpha^3} - \frac{S(k)}{3u_\alpha^3} - \frac{A}{\sqrt{k}\,u_\alpha^4} \\ &> 2k^2 \frac{q(k)}{u_\alpha^3} - \frac{A}{\sqrt{k}\,u_\alpha^4} \end{aligned} \tag{99}$$

with $$q(k) = E(k) - \tfrac{2}{3}[(1+k^2)\,E(k) - (1-k^2)\,K(k)]/k^2. \tag{100}$$

The expression

$$3q(k) = -2\int_0^{\frac{1}{2}\pi} \frac{1-\sin^2\varphi}{(1-k^2\sin^2\varphi)^{\frac{1}{2}}}\,\mathrm{d}\varphi + \int_0^{\frac{1}{2}\pi}(1-k^2\sin^2\varphi)^{\frac{1}{2}}\,\mathrm{d}\varphi \tag{101}$$

shows that $q(k)$ decreases monotonically from $q(0) = 0$ to $q(1) = -\frac{1}{3}$.

We now write $$\left.\begin{aligned} r_\beta - r_\alpha &> r_b - r_\alpha = -2k/u_\alpha, \\ r_\beta - r_\alpha &< \hat{r}_b - r_\alpha = -2\hat{k}/u_\beta < -2k/u_\alpha + A u_\alpha^{-2}/\sqrt{k}, \end{aligned}\right\} \tag{102}$$

and, as in (96) $$\frac{1}{u_\beta} - \frac{1}{u_\alpha} = -\frac{u_\beta - u_\alpha}{u_\alpha u_\beta} = \frac{2E(k)}{u_\alpha^3} + \frac{A}{\sqrt{k}\,u_\alpha^4}. \tag{103}$$

We put these estimates into (99) to obtain

$$(1+\epsilon)\left\{\left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right)\right\} < -\frac{2kq(k)}{u_\alpha^3} + \frac{A}{\sqrt{k}\,u_\alpha^4}, \tag{104}$$

where $|\epsilon|$ is small and $|A|$ is bounded, depending only on $ku_\alpha^2 \gg 1$.

Repeating the entire procedure starting with (65), we are led to the reverse inequality, with k replaced by $\hat{k}$ on the right. Applying (82) we obtain (104) with the inequality reversed, and thus

$$\left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right) = -\frac{2kq(k)}{u_\alpha^3}(1+\epsilon) + \frac{A}{\sqrt{k}\,u_\alpha^4}. \tag{105}$$

We place this estimate back into (99) (and the corresponding expression with $\hat{k}$) to find, using (102) and (103),

$$\left|\epsilon\left\{\left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right)\right\}\right| < \frac{A}{\sqrt{k}\,u_\alpha^4}, \tag{106}$$

the ϵ being the one that appears in (104). We are led to the basic relation for an outgoing arc (on which $0 \leqslant \psi \leqslant \frac{1}{2}\pi$)

$$\left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right) = -\frac{2kq(k)}{u_\alpha^3} + \frac{A}{\sqrt{k}\,u_\alpha^4}, \tag{107}$$

with bounded $|A|$, depending only on $ku_\alpha^2 \gg 1$.

The case of a returning arc ($\psi \geqslant \frac{1}{2}\pi$) does not yield immediately to the same discussion, and it is necessary to distinguish the case $k \approx 1$. We note (figure 14) that the comparison Delaunay surface

$v(r)$ of curvature $H = -\frac{1}{2}u_\beta$ now lies *below* $u(r)$, and provides an upper rather than a lower bound for $r_\beta - r_\alpha$. To obtain a lower bound, we observe that since $1 \leqslant E(k) \leqslant \frac{1}{2}\pi$ and $E'(k) < 0$, there is (for large $|u_\beta|$) a unique positive solution $\hat{\tau}$ of

$$\hat{\tau}^2 + u_\beta \hat{\tau} + 2E(-1 - u_\beta r_\beta - \hat{\tau} r_\beta) = 0; \tag{108}$$

that is, there exists a unique Delaunay comparison surface $\hat{v}(r)$ through (r_β, u_β) with mean curvature $H = -\frac{1}{2}\hat{v}_a$, so that

$$\hat{\tau} = \hat{v}_a - u_\beta = -2E(\hat{k})/\hat{v}_a, \tag{109}$$

with

$$\hat{k} = \frac{r_\beta - \hat{r}_a}{r_\beta + \hat{r}_a} = 1 + \hat{r}_a \hat{v}_a = -1 - r_\beta \hat{v}_a. \tag{110}$$

The solution curve $u(r)$ satisfies $u_r(r) > \hat{v}_r(r)$, $u(r) < \hat{v}(r)$, and hence $u(r)$ can be continued from r_β through decreasing r at least to the value $\hat{r}_a$. Letting $v(r)$ denote now the Delaunay surface through (r_β, u_β) with mean curvature $H = -\frac{1}{2}u_\beta$, we find $u_r(r) < v_r(r)$, $u(r) > v(r)$; it follows $u(r)$ cannot be continued to the minimum value r_a of definition of $v(r)$.

The relations analogous to (62, 65) become

$$\tfrac{1}{2}r_\beta^2 u_\beta + r_\beta < \tfrac{1}{2}r_\alpha^2 v_a + r_\alpha + \frac{1}{3u_\beta^3} S(k), \tag{111}$$

$$\tfrac{1}{2}r_\beta^2 u_\beta + r_\beta > \tfrac{1}{2}\hat{r}_a^2 \hat{v}_a + r_\alpha + \frac{1}{3\hat{v}_a^3} S(\hat{k}) - \tfrac{1}{2}u_\alpha(\hat{r}_a^2 - r_\alpha^2). \tag{112}$$

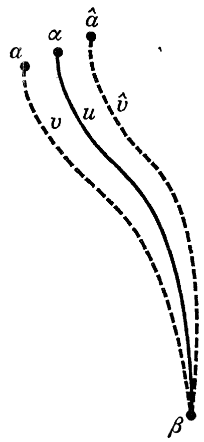

FIGURE 14. Comparison with Delaunay arcs; returning case.

As before, we may rewrite these relations:

$$\tfrac{1}{2}u_\beta\left[\left(r_\beta + \frac{1}{u_\beta}\right)^2 - \left(r_\alpha + \frac{1}{u_\alpha}\right)^2\right] < \frac{u_\alpha - u_\beta}{2}\left\{\frac{1}{u_\alpha u_\beta} + \frac{(1 + r_\alpha u_\alpha)^2}{u_\alpha^2}\right\} + \frac{1}{3u_\alpha^2} S(k) + \tfrac{1}{2}r_a^2(v_a - u_\alpha), \tag{113}$$

$$\tfrac{1}{2}u_\beta\left[\left(r_\beta + \frac{1}{u_\beta}\right)^2 - \left(r_\alpha + \frac{1}{u_\alpha}\right)^2\right] > \frac{u_\alpha - u_\beta}{2}\left\{\frac{1}{u_\alpha u_\beta} + \frac{(1 + r_\alpha u_\alpha)^2}{u_\alpha^2}\right\} + \frac{1}{3\hat{v}_a^3} S(\hat{k}) + \tfrac{1}{2}\hat{r}_a^2(\hat{v}_a - u_\alpha). \tag{114}$$

The further estimates must proceed differently, at least in the range $k \approx 1$.

From the defining relation (108) for $\hat{\tau}$ and the analogous one for $\tau = v_a - u_\beta$, follow $\tau, \hat{\tau} < A|u_\beta|^{-1}$ for large $|u_\beta|$. From

$$\hat{r}_a - r_a = \frac{-4E(\hat{k})}{u_\beta(u_\beta + \hat{\tau})^2} \tag{115}$$

thus follows

$$0 < \hat{r}_a - r_a = A|u_\beta|^{-3} \tag{116}$$

with bounded A.

Let $u^* = u(\hat{r}_a)$. For given $\lambda > 0$, consider a rectangle R of width $A|u_\beta|^{-3}$ and height $\lambda A|u_\beta|^{-3}$ as in figure 15. Since $u(r)$ cannot be extended to $r = r_a$, there must be at least one point (r_p, u_p) in R at which $|\tan\psi| > \lambda$, i.e. at which

$$|\cos\psi| < \frac{1}{(1+\lambda^2)^{\frac{1}{2}}}, \quad \sin\psi > \frac{\lambda}{(1+\lambda^2)^{\frac{1}{2}}}. \tag{117}$$

From (71), which holds also on a returning arc, we find, for all $r \leqslant \hat{r}_a$,

$$(\cos\psi)_u > u + (\sin\psi)/\hat{r}_a, \tag{118}$$

and hence, at the given point,

$$(\cos\psi)_u > u + \frac{\lambda}{\hat{r}_a(1+\lambda^2)^{\frac{1}{2}}} = u - \frac{1}{1-\hat{k}}\frac{\lambda}{(1+\lambda^2)^{\frac{1}{2}}}\left(u_\beta - \frac{2E(\hat{k})}{\hat{v}_a}\right). \tag{119}$$

We note $E(\hat{k}) < \frac{1}{2}\pi$; for any given k_0, $0 < k_0 < 1$, we choose λ so that

$$\frac{1}{1-k_0}\frac{\lambda}{(1+\lambda^2)^{\frac{1}{2}}} > 1. \tag{120}$$

For all sufficiently large $|u_\beta|$, the right side of (119) will then be positive for all $u \geqslant u_\beta$, for any $\hat{k}$ in $k_0 < \hat{k} < 1$. Thus, $\cos\psi$ is increasing (from a negative value) at $r = r_p$, and we conclude that (120), and hence also (119), continue to hold for all $r < r_p$ to which $u(r)$ can be continued. Integrating (119), we find that a vertical must appear within a height change

$$\delta^* u < A(1-\hat{k})/u_\beta, \quad \delta^* u = u_\alpha - u^*, \tag{121}$$

uniformly in $k_0 < \hat{k} < 1$.

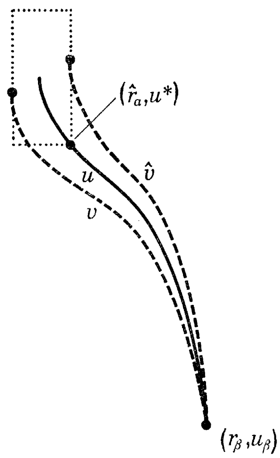

FIGURE 15. Estimate for u_α when $k \approx 1$.

For given $k < 1$ and large $|u|$ we may improve this result by estimating $\cos\psi^*$ explicitly. We have

$$r_\beta - \hat{r}_a \sin\psi^* = -\int_{\hat{r}_a}^{r_\beta} \rho u \,\mathrm{d}u < -\tfrac{1}{2}u_\beta(r_\beta^2 - \hat{r}_a^2), \tag{122}$$

$$r_\beta - \hat{r}_a = -\tfrac{1}{2}\hat{v}_a(r_\beta^2 - \hat{r}_a^2), \tag{123}$$

from which

$$\hat{r}_a(1-\sin\psi^*) < \tfrac{1}{2}(\hat{v}_a - u_\beta)(r_\beta + \hat{r}_a)(r_\beta - \hat{r}_a), \tag{124}$$

so that

$$1 - \sin\psi^* < \frac{4E(\hat{k})}{\hat{v}_a^2}\frac{\hat{k}}{1-\hat{k}}, \tag{125}$$

and hence

$$\cos\psi^* < \frac{2\sqrt{2}[E(\hat{k})]^{\frac{1}{2}}}{-\hat{v}_a}\left(\frac{\hat{k}}{1-\hat{k}}\right)^{\frac{1}{2}}. \tag{126}$$

We note (126) is similar to (70), however the term $(1+k)$ of (70) is replaced here by $(1-\hat{k})$. That is the reason the range $k \approx 1$ requires special consideration on a returning arc.

Repeating now the reasoning that led to (121), with (117) replaced by (126), leads to

$$\delta^* u < A/\sqrt{k}\, u_\beta^2 \tag{127}$$

for all $\hat{k} \leqslant k_0 < 1$. This estimate holds for all sufficiently large $|u_\beta|$.

A returning arc has in all cases exactly one inflexion between the vertical points (III iii). It is obvious a returning arc meets the hyperbola $ru = -1$ in exactly one point.

We proceed to obtain further estimates for $k \approx 1$, analogous to (79)–(90).

We have
$$k = \frac{r_\beta - r_a}{r_\beta + r_a}, \quad \hat{k} = \frac{r_\beta - \hat{r}_a}{r_\beta + \hat{r}_a}, \tag{128}$$

and thus
$$0 < k - \hat{k} = \tfrac{1}{2} r_\beta u_\beta \hat{v}_a (\hat{r}_a - r_a) < A u_\beta^{-2}, \tag{129}$$

by (115). We note that the factor $(1-k)$ of (82) no longer appears.

The estimate for $\hat{v}_a - v(\hat{r}_a)$ is complicated by the strong dependence on k of the position of the inflexion on $v(r)$. We avoid this difficulty by noting that the hemispherical surface $w(r)$ of constant mean curvature r_β^{-1}, that passes through (r_β, u_β), has larger mean curvature than does $v(r)$, hence $v(r) - w(r) > 0$. It follows that

$$\begin{aligned}\hat{v}_a - v(\hat{r}_a) &< (\hat{v}_a - w(0)) + (w(0) - w(\hat{r}_a)) \\ &= (\hat{v}_a - u_\beta - r_\beta) + [r_\beta - (r_\beta^2 - \hat{r}_a^2)^{\frac{1}{2}}] \\ &= -\frac{2E(\hat{k})}{\hat{v}_a} - r_\beta[1 - (\hat{r}_a^2/r_\beta^2)]^{\frac{1}{2}}.\end{aligned} \tag{130}$$

Formal estimation gives
$$E(k) \sim 1 - \tfrac{1}{4}(1-k^2)\,[\ln\{\tfrac{1}{16}(1-k^2)\} + 1], \tag{131}$$

for k near 1. Further,
$$\frac{1}{\hat{v}_a} \sim \frac{1}{u_\beta} + \frac{2E(\hat{k})}{u_\beta^3}, \tag{132}$$

and thus
$$\begin{aligned}\hat{v}_a - v(\hat{r}_a) &\lesssim -\frac{2E(\hat{k}) + r_\beta u_\beta}{u_\beta} + \frac{1}{2}\left(\frac{1-\hat{k}}{1+\hat{k}}\right)^2 r_\beta \\ &< \frac{A(1-\hat{k})\,|\ln(1-\hat{k})|}{u_\beta}\end{aligned} \tag{133}$$

with bounded $|A|$, uniformly in k for large $|u_\beta|$.

A repetition of previous procedures, with (128)–(133) in place of (79)–(90), leads after some calculation to
$$\begin{aligned}\frac{2\hat{k}q(\hat{k})}{u_\beta^3} + \frac{A(1-\hat{k})\ln(1-\hat{k})}{u_\beta^3} &< \left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right) \\ &< \frac{2kq(k)}{u_\beta^3} + \frac{B(1-\hat{k})\ln(1-\hat{k})}{u_\beta^3},\end{aligned} \tag{134}$$

with $|A|$ and $|B|$ bounded uniformly in k for large $|u_\beta|$. A formal, if tedious, calculation, based on asymptotic estimates for E and K for $k \approx 1$, yields

$$|q(k) - q(\hat{k})| < A u_\beta^{-2}. \tag{135}$$

We place this estimate and (129) into (134) to obtain the basic estimate, for a returning arc with $k \approx 1$,

$$\left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right) = \frac{2kq(k)}{u_\beta^3} + A\,\frac{(1-\hat{k})\ln(1-\hat{k})}{u_\beta^3}. \tag{136}$$

If, for some fixed k_0, there holds $0 < k \leqslant k_0 < 1$, $ku_\beta^2 \gg 1$, then the same procedure, using (127) in place of (121), yields for large $|u_\beta|$

$$\left(r_\alpha + \frac{1}{u_\alpha}\right) + \left(r_\beta + \frac{1}{u_\beta}\right) = \frac{2k\,q(k)}{u_\beta^3} + \frac{A}{\sqrt{k}\,u_\beta^4}, \tag{137}$$

with $|A| < A_0(k_0)$.

We summarize the information obtained thus far:

Theorem 8. (i) *A solution vertical at* (r_α, u_α), *such that* $r_\alpha u_\alpha > -1$ *and* (75) *holds with* $k = 1 + r_\alpha u_\alpha$, *will again become vertical at* (r_β, u_β), *with* $r_\beta u_\beta < -1$. *Between the two verticals there holds* $0 < \psi < \frac{1}{2}\pi$. *The height change is estimated by*

$$u_\beta = u_\alpha - 2E(k)/u_\alpha + \epsilon, \tag{138}$$

with
$$|\epsilon| < A/\sqrt{k}u_\alpha^2. \tag{139}$$

The solution arc meets the hyperbola $ru = -1$ *in exactly one point. The change in horizontal distance to the hyperbola at the two vertical points is controlled by* (107).

(ii) *Let* $k = -1 - r_\beta u_\beta$, *let*

$$\hat{k} = -1 - r_\beta \hat{v}_\alpha = -1 - r_\beta[u_\beta - 2E(\hat{k})/(u_\beta + \hat{r})] \tag{140}$$

(cf. (109)). *A solution vertical at* (r_β, u_β), *such that* $r_\beta u_\beta < -1$ *and* $ku_\beta^2 \gg 1$, *will again become vertical at* (r_α, u_α) *with* $r_\alpha u_\alpha > -1$. *Between the two verticals there holds* $\frac{1}{2}\pi < \psi < \pi$. *There is exactly one inflexion and one intersection with* $ru = -1$. *The height change is estimated by*

$$u_\alpha = u_\beta - 2E(\hat{k})/u_\beta + \epsilon, \tag{141}$$

with
$$|\epsilon| < A(1-\hat{k})/|u_\beta|, \tag{142}$$

and $A < A_0(k_0) < \infty$ *in any range* $0 < k_0 \leqslant k < 1$. *The change in horizontal distance to the hyperbola* $ru = -1$ *is controlled by* (136).

In any range $0 < k \leqslant k_0 < 1$, *if* $ku_\beta^2 \gg 1$, *then the solution will again become vertical at* (r_α, u_α), $r_\alpha u_\alpha > -1$; *the height change is again estimated by* (141), *but with*

$$|\epsilon| < A/\sqrt{k}\,u_\beta^2 \tag{143}$$

in place of (142). *The change in horizontal distance to the hyperbola* $ru = -1$ *is controlled by* (137).

VII. Asymptotic estimates

The results of § VI show that for large $|u|$, the solution curve contracts toward the hyperbola $ru = -1$ between any two successive verticals. The estimates (107), (136) and (137) contain quantitative information, which we now proceed to integrate to obtain new global information on the behaviour of the solution, when $|u_0|$ is large. We set $r_0 = 0$, denote the successive vertical points by (r_j, u_j), and write

$$\left.\begin{aligned} c_j &= |r_j + 1/u_j|, \quad k_j = -c_j u_j, \quad \delta c_j = c_{j+1} - c_j, \\ \hat{r}_j &= -2E(\hat{k}_j)/(u_j + \hat{r}_j), \quad \hat{k}_j = |1 + r_j(u_j + \hat{r}_j)|. \end{aligned}\right\} \tag{144}$$

Using (144), we find that (107) and (136) now take the form, for k, $\hat{k} \approx 1$,

$$\delta c_j = -\frac{2k_j\,q(k_j)}{u_j^3} + A_j \frac{1}{\sqrt{k_j}\,u_j^4} \quad \text{for} \quad j \text{ even,} \tag{145}$$

$$\delta c_j = -\frac{2k_j\,q(k_j)}{u_j^3} + A_j \frac{(1-\hat{k}_j)\ln(1-\hat{k}_j)}{u_j^3} \quad \text{for} \quad j \text{ odd,} \tag{146}$$

with $|A_j| < A < \infty$, uniformly for all sufficiently large $|u_j|$, in any range $0 < k_0 \leqslant k < 1$.

We are interested in (145, 146) for large $|u|$. We note $-q(k) \leqslant -q(1) = \frac{1}{3}$, $E(k) \leqslant \frac{1}{2}\pi$, and choose $u_{m_1}^2$ to be the (unique) solution of the equation

$$-\frac{4\pi}{u^2}\ln\left(\frac{4\pi}{u^2}\right) = \frac{1}{6A}. \tag{147}$$

Let $$k^{(1)} = \max\{k: -2kq(k) \leqslant -2A(1-k+4\pi/u_{m_1}^2)\ln(1-k+4\pi/u_{m_1}^2)\}. \tag{148}$$

Clearly, $0 < k^{(1)} < 1$, and

$$k^{(1)} > \max\{k: -2kq(k) \leqslant -2A(1-\hat{k})\ln(1-\hat{k})\}. \tag{149}$$

For all $k_j > k^{(1)}$, there holds $-A(1-\hat{k}_j)\ln(1-\hat{k}_j) < -k_j\,q(k_j)$.

Now choose u_{m_2} so that $(k^{(1)})^3 q^2(k^{(1)})\,u_{m_2}^2 > A^2$. For values

$$u_j^2 > \max\{u_{m_1}^2, u_{m_2}^2\} \tag{150}$$

we may write, since $k_j = -c_j u_j$, $$\delta c_j = -P_j c_j^3, \tag{151}$$

with $$\min_{k \geqslant k^{(1)}}\left(\frac{|q(k)|}{k^2}\right) < P_j < \max_{k \geqslant k^{(1)}}\left(\frac{3|q(k)|}{k^2}\right). \tag{152}$$

Integration of (151) in a range for which $k \geqslant k^{(1)}$, with $c_0 = -u_0^{-1}$, yields

$$2NP \sim c_N^{-2} - u_0^2, \tag{153}$$

for some P in the range indicated by (152).

We consider also the relation, which follows from (141)–(143),

$$\delta u_j = -\Lambda_j u_j^{-1}, \quad 2 \gtrsim \Lambda_j \gtrsim \pi, \tag{154}$$

and which integrates to $$u_N^2 \sim u_0^2 - 2\Lambda N, \quad 2 \gtrsim \Lambda \gtrsim \pi. \tag{155}$$

From (153) and (154) we calculate

$$k_N^2 \sim \frac{pu_N^2}{(1+p)\,u_0^2 - u_N^2}, \quad p = \Lambda P^{-1}, \tag{156}$$

and setting $u_N^2 = (1-\eta)\,u_0^2$, $$k_N^2 \sim \frac{p(1-\eta)}{p+\eta}. \tag{157}$$

Given $k^{(0)}$, $k^{(1)} < k^{(0)} < 1$, there will be, for all sufficiently large $|u_0|$, a unique smallest $N = N^{(1)}$ for which

$$k^{(1)} < \left[\frac{p(1-\eta)}{p+\eta}\right]^{\frac{1}{2}} < k^{(0)}; \tag{158}$$

the value of the expression in (158) tends to $k^{(0)}$ with increasing $|u_0|$.

We reformulate our result slightly, and summarize the information thus far obtained. We note that any set of points k_j = const lies on the hyperbola $1+ru$ = const, and that the singular solution $U(r)$ is asymptotic to the hyperbola $ru = -1$, as $r \to 0$. The following result holds for all $|u_0|$ sufficiently large.

THEOREM 9. *Given any* $k^{(0)}$, $k^{(1)} < k^{(0)} < 1$, *there exists* $\eta(k^{(0)}) > 0$ *such that the solution curve, starting at* $(0, u_0)$, *'separates' from the axis* $r = 0$ *and from the hyperbola* $ru = -2$, *after an interval* $u_N - u_0 = [1-(1-\eta)^{\frac{1}{2}}]\,|u_0|$, *in the sense that near the height* u_N *all points on the curve lie between the hyperbolas* $ru = -1 \pm k^{(0)}$. *Between* u_0 *and* u_N *a number* $N^{(1)} \sim \frac{1}{2}\eta u_0^2/A$ *of vertical points appear, and each vertical point is followed by another (on the opposite side of* $ru = -1$) *at a height change* $\delta u_j \sim -Au_j^{-1}$.

To proceed further, we return to the relations (107), (136) and (137); since $(1-k^{(0)}) \neq 0$ ($k^{(0)}$ independent of u_0), we may use (143) to write (145) and (146), for $j > N^{(1)}$, in the common form

$$\delta c_j = -\frac{2k_j\, q(k_j)}{u_j^3} + \frac{A_j}{\sqrt{k_j}\, u_j^4}. \tag{159}$$

We consider an interval in which the last two terms on the right in (122) will be small in relation to the first term. Since

$$\lim_{k\to 0} \frac{kq(k)}{k^3} = -\tfrac{1}{16}\pi, \tag{160}$$

the condition takes the form

$$k > A|u|^{-\frac{2}{7}} \tag{161}$$

for suitable A. Integration of (159) and of (154) yields, as above,

$$k_N \sim \left(\frac{p}{1+p}\right)^{\frac{1}{2}} \frac{u_N}{u_0}, \tag{162}$$

so that (161) now reads

$$|u_N| > A|u_0|^{\frac{7}{9}}. \tag{163}$$

We can in fact achieve the situation

$$\left.\begin{aligned} k &\sim A|u|^{-\frac{2}{7}}, \\ |u| &\sim A|u_0|^{\frac{7}{9}}, \end{aligned}\right. \tag{164}$$

for suitably large A (independent of u_0), asymptotically for large $|u_0|$, in a number $N^{(2)} \sim \frac{1}{2}u_0^2/A$ steps. In this configuration, *the solution curve has 'contracted' towards the hyperbola* $ru = -1$; *we compute in fact from* (153, 155)

$$\frac{c_{N^{(2)}}}{c_0} \sim \left(\frac{A}{A+P}\right)^{\frac{1}{2}} \tag{165}$$

as $|u_0| \to \infty$.

At the level $u_{N^{(2)}}$ the relation (159) no longer ensures a contraction at each step. The conditions for appearance of successive vertical points are, however, still satisfied, and (159) still suffices to bound the change δc_j at each step.

Let α satisfy $\frac{23}{9} < \alpha \leqslant 3$. In any range $A|u|^{-\frac{2}{7}} \geqslant k \geqslant B|u|^{(2\alpha-8)/(2\alpha+1)}$, we find

$$1/\sqrt{k}u_\alpha^4 < Ac^\alpha, \tag{166}$$

and we consider the inequality

$$|\delta c_j| < Ac^\alpha. \tag{167}$$

We integrate and simplify, noting that

$$c_{N^{(2)}}^{1-\alpha} \gg u_{N^{(2)}}^2 \tag{168}$$

for large $|u_0|$, to obtain

$$\left.\begin{aligned} A &< c_N/c_0 < B, \\ |u_N| &> |u_0|^{(2\alpha+1)/9}, \\ k_N &> |u_0|^{(2\alpha-8)/9}; \end{aligned}\right\} \tag{169}$$

thus the solution remains in a strip of sensibly constant width about $ru = -1$, until a height

$$|u_{N^{(3)}}| \sim |u_0|^{(2\alpha+1)/9}.$$

We conclude in particular the existence of a constant A such that *in an interval*

$$|u_0|^{(2\alpha+1)/9} < |u| < 2|u_0|^{(2\alpha+1)/9} \tag{170}$$

there holds $k_N < A|u|^{(2\alpha-8)/(2\alpha+1)}$. We assert that *for all sufficiently large* $|u|$, *the solution curve lies interior to a strip determined by*

$$k = -cu = (A+1)|u|^{-\frac{2}{7}} \tag{171}$$

uniformly in u_0 *as* $|u_0| \to \infty$, *for any* A *sufficiently large to justify* (164). This is clearly the case in the interval (170). If the curve $u(r)$ were to extend outside the strip (171) for value of u exceeding $-|u_0|^{(2\alpha+1)/9}$, there must be a first point p on the boundary of the strip. By comparison with Delaunay surfaces through the point p, one then sees (note either the condition (75) or the corresponding condition with u_β is satisfied at p) that a vertical would appear on or outside the strip $k = A|u|^{-\frac{2}{7}}$. Let q_1 be the first such point. The estimate (159), applied now in the direction of increasing $|u|$, shows that a preceding vertical can be found at a point q_0, with horizontal distance to $ru = -1$ exceeding that from q_1. The strip $k = A|u|^{-\frac{2}{7}}$ is however narrower at q_0 than at q_1. This contradiction establishes the assertion.

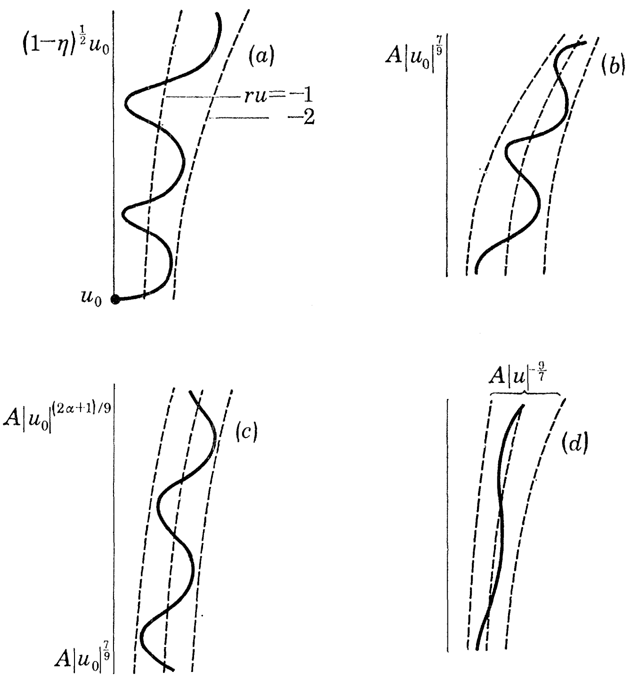

FIGURE 16. Asymptotic behaviour for large $|u_0|$ at four levels (scales differ). (*a*) Initial separation from axis, (*b*) contraction toward hyperbola, (*c*) confinement to strip of constant width, (*d*) behaviour far from $|u_0|$.

We summarize:

THEOREM 10. *Given* $k^{(0)} > k^{(1)}$, $|u_0|$ *large, there is an* $\eta(k^{(0)})$ (*determined by* (157) *and* (158)) *so that* $k < k^{(0)}$ *for* $|u_N| < |u_{N^{(1)}}| = (1-\eta)^{\frac{1}{2}}|u_0|$. *The curve can be continued through successive verticals to a height* $|u_{N^{(2)}}| \sim A|u_0|^{\frac{7}{9}}$, *for suitably large* A, *at which level it has contracted towards* $ru = -1$ *in a ratio given by* (165). *For any* α, $\frac{23}{9} < \alpha < 3$, *the curve can be continued further through successive verticals until a height* $|u_{N^{(3)}}| \sim |u_0|^{(2\alpha+1)/9}$, *and is confined to a strip of sensibly constant width, as indicated by* (169). *For smaller values of* $|u|$ (*relative to* $|u_0|$) *vertical points presumably cease to appear; however the curve lies within a strip* *about $ru = -1$, of width determined by $k = A|u|^{-\frac{2}{7}}$, for sufficiently large A (independent of u_0), Specifically, there exists A such that for any fixed (sufficiently large) $\hat{u}$, there holds, for $(\hat{r}, \hat{u})$ on the solution curve,*

$$|\hat{r} - 1/\hat{u}| < A|\hat{u}|^{-\frac{9}{7}} \tag{172}$$

uniformly in u_0, as $|u_0| \to \infty$.

The global asymptotic behaviour is sketched in figure 16.

VIII. A compactness property

Let us consider the family of solution curves, represented in the form $r = f(u; u_0)$, with u_0 as parameter, $|u_0| \to \infty$. The result (172) shows that for large $|u|$ the curve is confined to a narrow strip about $ru = -1$, and the method of proof of (172) yields as corollary the existence of a constant A such that on any fixed interval $a \leqslant u \leqslant b$,

$$|\partial f/\partial u| < A < \infty,$$

for all sufficiently large $|u_0|$.

It follows there is a subsequence of values $u_0 \to -\infty$ such that the corresponding functions $f(u; u_0)$ converge, uniformly on compact intervals, for all $|u|$ sufficiently large that (172) applies. The limit curve $\mathscr{C}$: $r = \mathscr{F}(u)$, when described with arc length as parameter, yields a solution of the parametric system (3). There holds

$$|1 + u\mathscr{F}(u)| < A|u|^{-\frac{2}{7}} \tag{173}$$

for all large $|u|$.

Each of the curves $f(u; u_0)$ can be extended globally without self-intersection as indicated in theorem 6. Applying the general continuous dependence theorem, we find that the limit curve $\mathscr{C}$ has the same property (a reasoning similar to the proof of theorem 6 excludes self-intersection). The curve $\mathscr{C}$ has the asymptotic property $u\mathscr{F}(u) \sim -1$ for large $|u|$, and the oscillatory behaviour indicated in figure 1 for large r. It seems likely the curve $\mathscr{C}$ is the singular solution $U(r)$, and we conjecture that is the case.

IX. Isolated character of global solutions

There is strong numerical evidence to suggest that global solutions of (3) lying in the region between the envelope of the solutions $r = f(u; u_0)$ and the u axis, and without limit sets or self-intersections, are rare in the manifold of all solutions. We know of no such solutions, apart from those described in this paper and the singular solution $U(r)$. In figure 17 are shown samples of the result of numerical integration of (3) through the initial point p determined by the first intersection of $f(u; -8)$ with $U(r)$ and for varying initial angles α, measured counterclockwise from the arc of the curve $f(u; -8)$ emanating from p in the direction of increasing u. The various curves are thus distinguished by their directions at the point p, measured relative to that of the curve $f(u; -8)$ at that point.

We note the curves $f(u; u_0)$ can apparently be extended below the level $u = u_0$, if the isolated (singular) point of contact with the u axis is admitted. The point appears to mark a transition in qualitative behaviour; above it, the curve behaves like a Delaunay arc generated by an ellipse (§ III). Below that point, the curve has the general appearance of a Delaunay arc generated by a hyperbola, with the characteristic double points of those arcs.

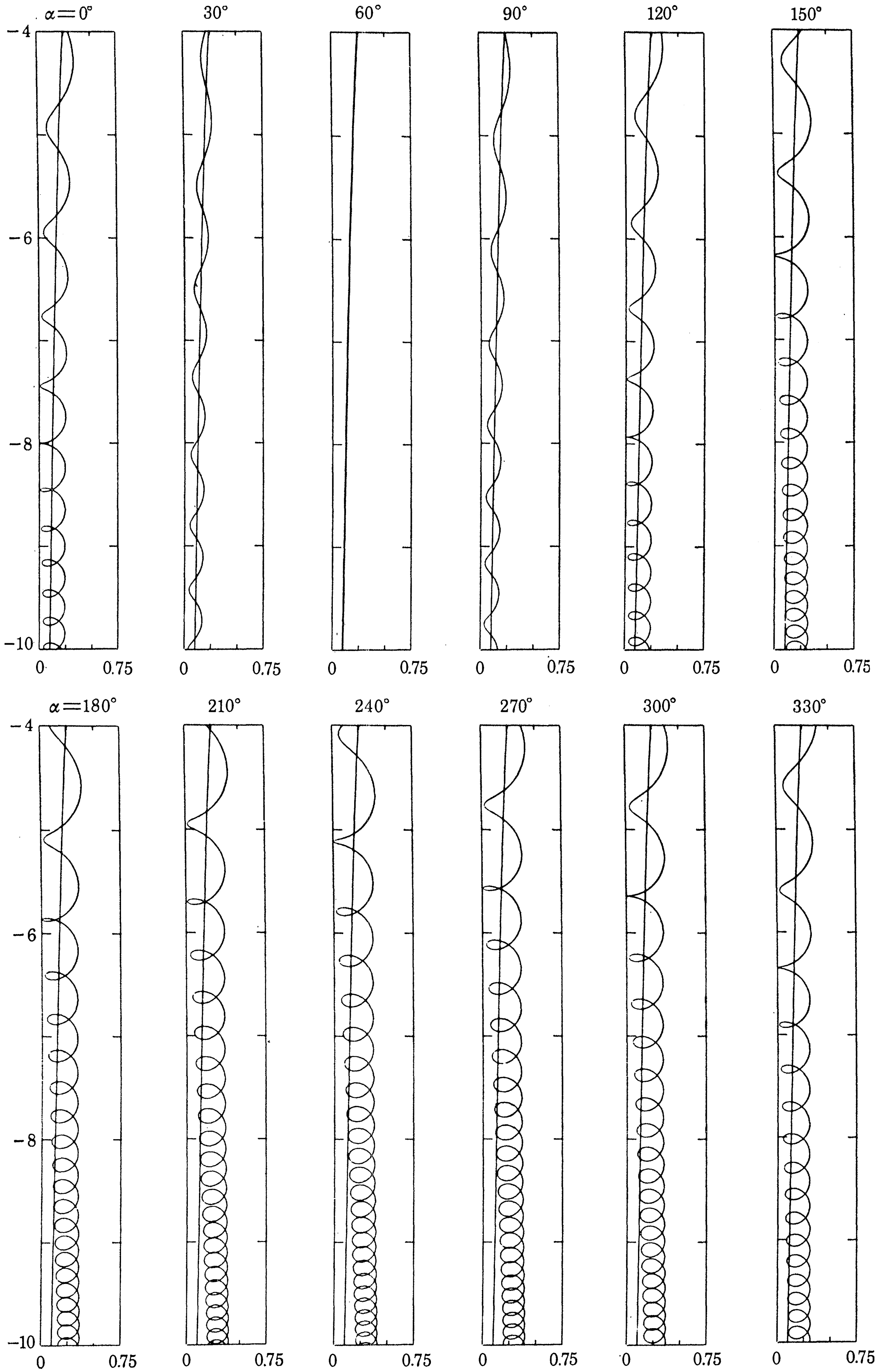

FIGURE 17. Singular solution and other solutions of (3).

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An analogous transition occurs on neighbouring solutions without occurrence of singular points on the axis. In any event, if such singular points are admitted, the corresponding (extended) curves $f(u\colon u_0)$ are embedded naturally in a solution set, all of which develop double points for sufficiently negative u, with (we conjecture) the single exception of the solution $U(r)$.

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Appendix

We prove here the local existence theorem discussed in the Introduction.

Theorem. *For any prescribed* $u_0 < 0$, *there is a unique solution* $u(r; u_0)$ *of* (2) *in an interval* $0 < r < \delta(u_0)$, *such that* $\lim_{r\to 0} u(r; u_0) = u_0$.

Proof. We introduce (formally) the angle ψ as independent variable, obtaining the equations

$$\frac{\mathrm{d}r}{\mathrm{d}\psi} = \frac{-r\cos\psi}{\Delta}, \quad \frac{\mathrm{d}u}{\mathrm{d}\psi} = \frac{-r\sin\psi}{\Delta}, \tag{174a, b}$$

with $\Delta = ru + \sin\psi$. Writing $r^2 = v$, we derive the system

$$\mathrm{d}\Delta^2/\mathrm{d}\psi = 2(\cos\psi - v)\sin\psi, \tag{175}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\psi} = \frac{2v\cos\psi}{\Delta}, \tag{176}$$

in which u does not appear explicitly.

Corresponding to $\delta(u_0) > 0$ (to be determined) we introduce a Banach space $\mathscr{B}$ of pairs $(\xi(\psi), \eta(\psi))$ of continuous functions on $\mathscr{I}_\delta$: $0 \leqslant \psi \leqslant \delta$, with norm defined by

$$\|(\xi, \eta)\| = \max_{\mathscr{I}_\delta} |\xi(\psi)| + \max_{\mathscr{I}_\delta} |\eta(\psi)|, \tag{177}$$

and we define a closed convex set $\mathscr{M} \subset \mathscr{B}$ by the relations

$$|\xi| \leqslant 16u_0^{-4}\sin^4\psi, \tag{178}$$

$$|\eta| \leqslant 3u_0^{-2}\sin^3\psi, \tag{179}$$

on $\mathscr{I}_\delta$. A continuous mapping $\mathscr{F}: (\xi, \eta) \to (\hat{\xi}, \hat{\eta})$ of $\mathscr{M} \to \mathscr{B}$ is determined by the relations

$$v = 4u_0^{-2}\sin^2\psi + \xi, \tag{180}$$

$$\Delta = -\sin\psi + \eta, \tag{181}$$

$$\hat{v} = 4u_0^{-2}\sin^2\psi + \hat{\xi}, \tag{182}$$

$$\hat{\Delta} = -\sin\psi + \hat{\eta}, \tag{183}$$

$$\frac{d\hat{v}}{d\psi} = \frac{2v\cos\psi}{\Delta} \quad \text{on} \quad \mathscr{I}_\delta - \{0\}, \tag{184}$$

$$\frac{d\hat{\Delta}^2}{d\psi} = 2(\cos\psi - v)\sin\psi \quad \text{on} \quad \mathscr{I}_\delta - \{0\}, \tag{185}$$

$$\hat{v}(0) = 0, \quad \lim_{\psi\to 0}(\hat{\Delta}/\sin\psi) = -1, \tag{186}$$

and one sees easily that if $\delta(u_0)$ is sufficiently small, then $\mathscr{F}: \mathscr{M} \to \mathscr{M}$.

We assert $\mathscr{F}(\mathscr{M})$ is precompact. For the intermediate mapping $(\xi, \eta) \to (\hat{v}, \hat{\Delta})$ determined by (180), (181) and (184)–(186) carries $\mathscr{M}$ onto a set of equicontinuous bounded functions, hence by (182) and (183) the image set $(\hat{\xi}, \hat{\eta})$ are also equicontinuous and bounded. We conclude from Schauder's fixed point thorem (*cf.* Gilbarg & Trudinger 1977, p. 222) that there exists a solution of (175) and (176) in $\mathscr{I}_\delta$, with $v(0) = 0$, $\lim_{\psi\to 0}(\Delta/\sin\psi) = -1$.

Using the functions v, Δ thus obtained, we define $r(\psi)$ to be the positive root of v, and we define $u(\psi)$ by integrating (174*b*), with $u(0) = u_0$. We shall show the three functions u, r, Δ determine a solution of (174*a*, *b*) with the requisite properties.

Lemma. *For the functions just obtained, there holds* $\Delta \equiv ru + \sin\psi$.

Proof. Set $w = \Delta - (ru + \sin\psi)$. Using (174) and (175) we find

$$dw/d\psi = -w(\cos\psi)/\Delta, \tag{187}$$

and thus
$$w(\psi) = w(\psi_1)\exp\left\{-\int_{\psi_1}^{\psi}\frac{\cos\psi}{\Delta}\,d\psi\right\}, \tag{188}$$
for $0 < \psi_1 < \psi < \delta(u_0)$.

From (179) and (181) we have, as $\psi \to 0$,

$$\Delta = -\sin\psi + O(\psi^3), \tag{189}$$

and thus
$$\exp\left\{-\int_{\psi_1}^{\psi}\frac{\cos\psi}{\Delta}\,d\psi\right\} = \frac{1}{\sin\psi_1} + O(\psi_1). \tag{190}$$

Similarly, using (174*b*) and (178)–(181),

$$r = -2u_0^{-1}\sin\psi + O(\psi^3), \tag{191}$$

and
$$u = u_0 + O(\psi^2), \tag{192}$$

and hence
$$ru + \sin\psi = -\sin\psi + O(\psi^3), \tag{193}$$

$$w = \Delta - (ru + \sin\psi) = O(\psi^3), \tag{194}$$

and it follows that the right side of (188) tends to zero with ψ_1. Thus, $w(\psi) \equiv 0$, as was to be shown.

We may now complete the proof. From (174*a*), (189) and (191) we find that $dr/d\psi \neq 0$ in $\mathscr{I}_\delta - \{0\}$ if δ is sufficiently small. Thus, we may write, using (174*a*) and the lemma,

$$\begin{aligned}(r\sin\psi)_r &\equiv \sin\psi + r(\sin\psi)_r \\ &= \sin\psi + r\cos\psi\, d\psi/dr \\ &= -ru,\end{aligned}$$

that is, (2) holds in the interval $0 < r < r(\delta)$, which was to be shown.

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References

Bakker, G. 1928 Kapillarität und Oberflächenspannung. *Handbuch der Experimentalphysik*, vol. VI (ed. W. Wien & F. Harms). Leipzig: Akademische Verlagsgesellschaft.

Bashforth, F. & Adams, J. C. 1883 *An attempt to test the theories of capillary action by comparing the theoretical and measured forms of drops of fluid*. Cambridge University Press.

Concus, P. 1968 Static menisci in a vertical right circular cylinder. *J. Fluid Mech.* **34**, 481–495.

Concus, P. & Finn, R. 1974 On capillary free surfaces in a gravitational field. *Acta Math.* **132**, 207–223.

Concus, P. & Finn, R. 1975*a* A singular solution of the capillary equation. I. Existence. *Invent. math.* **29**, 143–148.

Concus, P. & Finn, R. 1975*b* A singular solution of the capillary equation. II. Uniqueness. *Invent. math.* **29**, 149–160.

Delaunay, C. E. 1841 Sur la surface de révolution dont la courbure moyenne est constante. *J. Math. pures appl.* **6**, 309–315.

Finn, R. 1974 Capillarity phenomena. *Usp. mat. Nauk* **29**, 131–152.

Finn, R. 1976 On the behavior of a capillary surface near a singular point. *J. Analyse math.* **30**, 156–163.

Gilbarg, D. & Trudinger, N. S. 1977 *Elliptic partial differential equations of second order*. New York, Heidelberg, Berlin: Springer-Verlag.

Hartland, S. & Hartley, R. W. 1976 *Axisymmetric fluid–liquid interfaces*. New York: Elsevier.

Hida, K. & Miura, H. 1970 The shape of a bubble or a drop attached to a flat plate subject to a weak gravitational force. *J. Japan aeron. astron. astr. Soc.* **18**, **433**–440.

Jahnke, E. & Emde, F. 1945 *Tables of functions*. New York: Dover Publications.

Johnson, W. E. & Perko, L. M. 1968 Interior and exterior boundary value problems from the theory of the capillary tube. *Arch. ration. Mech. Analysis* **29**, 125–143.

Lohnstein, Th. 1891 Dissertation, University of Berlin.

Minkowski, H. 1907 Kapillarität. In *Encyklopedie mathematischen Wissenschaften*, vol. V (1), 558–613. Leipzig: Teubner.

Padday, J. F. 1971 The profiles of axially symmetric menisci. *Phil. Trans. R. Soc. Lond.* A **269**, 265–293.

Pitts, E. 1973 The stability of pendent liquid drops. I. Drops formed in a narrow gap. *J. Fluid Mech.* **59**, 753–767.

Pitts, E. 1974 The stability of pendent liquid drops. II. The axially symmetric case. *J. Fluid Mech.* **63**, 487–508.

Thomson, W. 1886 Capillary attraction. *Nature, Lond.* **34**, 270–272, 290–294, 366–369.

Wente, H. C. 1973 An existence theorem for surfaces in equilibrium satisfying a volume constraint. *Arch. ration. Mech. Analysis* **50**, 139–158.

Wente, H. C. 1978 The stability of the axially symmetric pendent drop. Preprint, Department of Mathematics University of Toledo, Ohio (to appear in *Pacific J. Math.*)

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